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# Compactly Embedded Cartan Algebras and Invariant Cones in Lie Algebras

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We shall consider finite dimensional real Lie algebras. A subset $W$ of a Lie algebra is a *cone* if it is closed topologically, additively, and under multiplication with nonnegative scalars. It is said to be *invariant* if it is further closed under all inner automorphisms of $L$. This is tantamount to saying that

$$e^{\operatorname{ad} x} W = W \qquad \text{for all } x \in L.$$

Let us look at some examples.

(1) Let $L$ be a compact Lie algebra with nontrivial center. Then we can write $L$ as a direct sum $\mathbb{R} \cdot c \oplus I$ with nonzero central element $c$ and an ideal $I$. The group $G$ of inner automorphisms of the compact Lie algebra $I$ is compact. Hence there is a compact convex $G$-invariant neighborhood $C$ of 0 in $I$. Then the set $W = \mathbb{R}^+ \cdot (c + C)$ is an invariant cone in $L$.

One knows that this construction pretty much exhausts all invariant cones in compact Lie algebras; this is literally correct for cones with inner points but without nonzero vector subspaces. (For details see, e.g., [HH86b].)

(2) Let $L = sl(2, \mathbb{R}) = so(2, 1)$. The Cartan–Killing form $B$ on $L$ has signature $+ + -$ and the set of all $x \in L$ with $B(x, x) \leqslant 0$ is a double cone $W \cup -W$ each of whose "halves" $W$ and $-W$ is an invariant cone. In fact, the vectors

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



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span a hyperbolic plane with respect to $B$, and $L$ is an orthogonal direct sum of this hyperbolic plane and the vector subspace spanned by the vector

$$H=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Each of the vectors $U$ and $H$ span Cartan subalgebras, but both are of an entirely different nature. The first one generates a circle group in the group of inner automorphisms; the vector $U$ is an inner point of one of the invariant cones, say $W$, while $H$ is in the complement of the double cone and generates a one parameter subgroup isomorphic to $\mathbb{R}$ in the group of inner automorphisms.

The explicit details of this example are very well understood (see, e.g., [HH85a]). It is the lowest dimensional example of a simple algebra $L$ which contains a pointed invariant cone $W$ with inner points (where we have called a cone *pointed* if it does not contain nonzero vector spaces). The simple Lie algebras supporting such cones and these cones are classified. This classification has been accomplished by Ol'shanskii and by Paneitz [Ol81], [Pa84] (see also [KR82]). Essential impulses to proceed in this direction are due to Vinberg [Vi80] who was the first one to contemplate the issue for simple Lie algebras. Invariant cones in simple Lie algebras occur in representation theory [Ol81] and in the study of holomorphic symmetric domains [Ro80].

(3) Let $E$ be a finite dimensional Hilbert space with respect to an inner product $(\cdot \mid \cdot)$. Assume that $E$ has even dimension and that $D\colon E \to E$ is a skew symmetric automorphism relative to this inner product which we may assume to have the matrix

$$\begin{pmatrix} 0 & 1 & \cdots & & 0\\ -1 & 0 & & & \\ \vdots & & \ddots & & \vdots\\ & & & 0 & 1\\ 0 & \cdots & & -1 & 0\end{pmatrix}$$

relative to a suitable orthinormal basis.

Now we define a Lie algebra $L=\mathbb{R}\times E\times\mathbb{R}$ as follows: Addition is componentwise, and the Lie bracket is given by

$$[(r, u, s), (r', u', s')] = (0, rDu' - r'Du, (u \mid Du')).$$

Furthermore, we define a quadratic form $Q$ on $L$ as follows:

$$Q((r, u, s), (r', u', s') = rs' + r's + (u \mid u').$$

Then $Q$ has signature $+ \cdots + -$; indeed $L$ is the $Q$-orthogonal sum of the hyperbolic plane $\mathbb{R} \times \{0\} \times \mathbb{R}$ and the Hilbert space $\{0\} \times E \times \{0\}$. Now $L$ is a solvable Lie algebra whose commutator algebra $L' = [L, L]$ is a Heisenberg algebra $\{0\} \times E \times \mathbb{R}$. The Lorentzian form $Q$ is invariant in the sense that $Q([x, y], z) = Q(x, [y, z])$. The set of all $x \in L$ with $Q(x, x) \leqslant 0$ is a double cone $W$ each of whose two "halves" $W$ and $-W$ is an invariant cone. The case $\dim E = 2$, i.e., $\dim L = 4$ is the so-called oscillator algebra; it is the Lie algebra of the observables of the quantum mechanical model of the harmonic oscillator. The discovery of the invariant Lorentzian cones in this algebra is due to Guts and Levichev [GL84]. The description of Lorentzian forms on the series of oscillator algebras was given by Medina and Revoy [MR83, MR84]. A detailed discussion of this example and the proof of appropriate uniqueness statements was also given in [HH85c]. In fact all three examples yield Lorentzian invariant cones in Lie algebras, and we showed in [HH85c] that forming the orthogonal direct sum of one of these examples with a compact algebra equipped with a positive definite invariant form yields all possible examples of Lie algebras with invariant Lorentzian forms.

The higher dimensional examples of simple Lie algebras with invariant cones give nonLorentzian instances of invariant cones.

The examples and the bibliography of the search for invariant cones in finite dimensional Lie algebras amphasize the need for a *unified theory of invariant cones in arbitrary Lie algebras*. We propose such a theory in this paper and a subsequent monograph [HHL89].

The first question one has to address is the question of the possible direction of a general theory of invariant cones and the meaning of a potential classification of all pairs $L$, $W$ of a Lie algebra $L$ together with an invariant cone $W$. A key idea already appears in the work of Vinberg who noted tht in the case of a simple algebra $L$ the Cartan algebras $H$ which meet the cone $W$ play a crucial role.

Cartan subalgebras are at the bottom of any general theory. They are commonly known in the context of semisimple algebras for their role in the classification, among other things. However, they are perfectly well defined in any finite dimensional Lie algebra. In the context of a general theory of invariant cones, however, a particular type of Cartan subalgebra turns out to be relevant. In fact we will show that every point in the interior of a pointed invariant cone in a Lie algebra lies in a Cartan subalgebra $H$ such that $e^{\operatorname{ad} H}$ is a relatively compact analytical subgroup of the group $Gl(L)$ of all vector space automorphisms of $L$. We shall call those Cartan algebras *compactly embedded*. We shall observe that they are abelian and all conjugate under inner automorphisms of $L$, and that they are each contained in a unique maximal compact subalgebra $K(H)$ of $L$. The invariant cones $W$ of a Lie algebra $L$ are uniquely determined by their intersection $W \cap H$

with any fixed compactly embedded Cartan subalgebra $H$. In other words, if $W_1$ and $W_2$ are pointed invariant cones with inner points such that $W_1 \cap H = W_2 \cap H$, then $W_1 = W_2$. This uniqueness theorem affords us the opportunity to attempt the classification of invariant cones $W$ in $L$ in terms of cones $C$ in a compactly embedded Cartan algebra $H$. We shall show which properties a cone $C$ in $H$ must have so that it is the intersection $H \cap W$ of $H$ with an invariant cone $W$ of $L$. This program requires a sequel to the present article in which we shall analyze the root decomposition of $L$ with regard to a compactly embedded Cartan algebra $H$ in detail (see [HHL89]).

In the present article, however, we have enough work with the technical execution of the program we have just outlined. The immediate Lie algebra theoretical difficulties start with the possibility that the group of inner automorphisms of $L$ which is generated by all automorhisms of the form $e^{\operatorname{ad} x}$, $x \in L$, may very well be a nonclosed analytical subgroup of $Gl(L)$. This possibility for which we have to make provisions is the cause for considerable technical complications which we have to deal with first. In order to recognize the situation, one simply has to consider the Lie algebra $L$ of all matrices

$$\begin{pmatrix} is & 0 & u \\ 0 & it & v \\ 0 & 0 & 0 \end{pmatrix}, \qquad s, t, u, v \in \mathbb{R} \quad \text{with } t = \sqrt{2}\, s.$$

Here $L$ is a perfectly innocent solvable metabelian Lie algebra with a faithful adjoint representation (due to the absence of nonzero central elements!) whose group of inner automorphisms is not closed. The algebra of all matrices with $u = v = 0$ is a compactly embedded Cartan algebra which generates a nonclosed group of inner automorphisms which is dense in a maximal torus of the closure of the group of inner automorphisms in $Gl(L)$. In fact, if $A$ is the group of inner automorphisms and $T$ this maximal torus, then $\bar{A} = AT$, and we shall see that this phenomenon is quite general. The preceding example, by the way, does not allow invariant pointed cones at all. In the example of the oscillator algebra a compactly embedded Cartan algebra is the two-dimensional algebra $\mathbb{R} \times \{0\} \times \mathbb{R}$ and it generates a one-dimensional circle group in $Gl(L)$. Whether the pathology of a nonclosed group of inner automorphisms actually occurs when $L$ supports an invariant pointed cone with inner points is unknown. Nevertheless, we have to build a substantial amount of Lie algebra theory involving the Lie group theory of dense analytic subgroups. A good deal of the results we develop in this effort is of independent interest since a priori it has nothing to do with invariant cones. A substantial portion of the general theory of compactly embedded Cartan algebras should likewise be of interest on its own.

The organization of our paper is as follows.

The first section is concerned with dense analytic subgroups of Lie groups. We show that for a dense analytic subgroup $A$ of a Lie group $G$ we have $G = AT$ for any maximal torus $T$. A fortiori, $G = AK$ for any maximal compact subgroup $K$. In particular, this shows that the maximal compact subgroups of $G$ are conjugate under elements of $A$. We apply this to prove a technically sounding result, which nevertheless is of central importance for what follows: If $\pi: G \to Gl(n)$ is a representation of a connected Lie group and $K$ is a maximal compact subgroup of $\overline{\pi(G)}$, and if $H$ is the arc component of the identity in $\pi^{-1}(K)$, then $H$ is an analytic subgroup of $G$ such that $\overline{\pi(H)} = K$. We then consider representations $\rho: L \to gl(n)$ of a Lie algebra and call a subalgebra $M$ of $L$ *$\rho$-compact* if $e^{\rho(M)}$ generates a relatively compact subgroup of $Gl(n)$. We develop a theory of such subalgebras in reasonable generality to allow, in particular, for applications to the adjoint representation of the Lie algebra $L$. A sequence of examples at the end of the section illustrates what has been discussed in the section.

The second section considers representations $\rho: L \to gl(n)$ and the associated $\rho$-compact elements of $L$. These are completely characterized. The main application is the adjoint representation of $L$. The set of ad-compact elements will be denoted $\operatorname{comp} L \subset L$. As important information we shall show that an element $x$ of $L$ is an interior point of $\operatorname{comp} L$ if and only if its centralizer is entirely contained in $\operatorname{comp} L$.

This paves the way for Section 3. Here we show that a regular element $x$ of $L$ is an inner point of $\operatorname{comp} L$ if and only if the Cartan algebra which it determines in the usual way is compactly embedded. This will first give more information on the set $\operatorname{comp} L$ and its interior, and it will finally lead to the conclusion that two compactly embedded Cartan algebras are always conjugate under inner automorphisms of $L$. (Much energy in the first two sections is spent on the fact that such conjugacy assertions can be spelled out in terms of the inner automorphisms of $L$ directly rather than in terms of automorphisms which may only be approximated by inner automorphisms.) We shall then lead up to a proof of the fact that every compactly embedded Cartan algebra $H$ of $L$ is contained in a uniquely determined maximal compactly embedded subalgebra $K(H)$ of $L$. All maximal compactly embedded subalgebras are again conjugate under inner automorphisms of $L$.

In the fourth section we bring this entire theory to bear on the question of invariant cones with the results which we have already explained and for whose details we have to refer the reader to the text itself.

## 1. Background Material on Lie Groups

1.1. Lemma. *Let $G$ be a connected Lie group and $G'$ its commutator subgroup. If $C(G)$ denotes the maximal compact connected subgroup of the center of $G$, then $G'C(G)$ is closed.*

*Proof.* (i) Let $Z$ denote the center of $G$ and $Z_0$ its identity component. We shall first show that $G'Z_0$ is closed. Indeed, since $Z$ is the kernel of the adjoint representation Ad, the factor group $G/Z$ has a faithful representation. Then $(G/Z)'$ is closed (see, e.g., [Hoch65, p. 224, Theorem 4.4]). But $(G/Z)' = G'Z/Z$, whence $G'Z$ is closed. The subgroup $G'Z_0$ is analytic as an arcwise connected subgroup of $G$. Let $(G'Z_0)_L$ denote this group with its intrinsic Lie group topology which is possibly finer than the induced topology of $G'Z_0$. The continuous homomorphism $gZ_0 \mapsto gZ$: $(G'Z_0)_L/Z_0 \to G'Z/Z$ is open by the open mapping theorem for finite dimensional connected Lie groups. It factors through the continuous homomorphism $(G'Z_0)_L/Z_0 \to G'Z_0/Z_0$ which we thus conclude to be open, too. This means $(G'Z_0)_L = G'Z_0$. Hence $G'Z_0$ is a Lie group and therefore closed.

(ii) We next consider the subgroup $A = G'C(G)$ which is analytic, since it is arcwise connected. We shall show that $A$ is closed. We have $\bar{A} = \overline{G'C(G)} \subseteq G'Z_0$ by part (i) above. Hence we note $L(G') + L(C) = L(A) \subseteq L(\bar{A}) \subseteq L(G') + L(Z)$ by the abbreviation $C = C(G)$. Our claim will be established if we can show $L(A) = L(\bar{A})$. We can write $L(\bar{A}) = L(A) \oplus V$ with a vector space complement of $L(G') + L(C) = L(A)$ contained in $L(Z)$. We have to show $V = (0)$. We assume $V \neq (0)$ and derive a contradiction. Since the center $Z$ is closed and $V \cap L(C) = (0)$ we conclude that $\exp V$ is a closed central subgroup isomorphic to $\mathbb{R}^n$ with $n = \dim V \geqslant 1$. We can factor an $n-1$-dimensional vector subgroup of $\exp V$ and derive a contradiction in the factor group. Therefore we may assume henceforth that $\dim V = 1$ and that $\exp V \cong \mathbb{R}$. We set $\exp V = R$ and note that $\bar{A} = AR$ and $A \cap R \neq R$ since $L(\bar{A}) = L(A) \oplus L(R)$ on account of $L(R) = V$. If $A_L$ denotes the group $A$ with its intrinsic Lie group topology, then the morphism $f\colon A_L \to \bar{A}/R$ given by $f(a) = AR$ induces an isomorphism on the Lie algebra level and has, therefore, a discrete kernel. A discrete normal subgroup in a connected Lie group is a finitely generated abelian group. But $\ker f = A \cap R$. The finitely generated subgroups of $R \cong \mathbb{R}$ are cyclic. In particular, $A \cap R$ is closed in $R$ and hence in $\bar{A}$. By the open mapping theorem, the morphism $f$ is open; since it factors through $A_L/(A \cap R) \to A/(A \cap R)$ we conclude, similarly to our argument in part (i), that this latter morphism is an isomorphism and that, as a consequence, $A/(A \cap R)$ is a Lie group. But, then, on the one hand, $A/(A \cap R)$ is dense in $\bar{A}/(A \cap R)$, while $\bar{A}/(A \cap R)$ is the direct product of the closed Lie subgroups

$A/(A \cap R)$ and $R/(A \cap R)$ on the other. This contradiction completes the proof. ▮

$$\begin{tikzcd}[column sep=small] & \bar{A} \arrow[dl, dash] \arrow[dr, dash] & \\ A \arrow[dr, dash] & & R \arrow[dl, dash] \\ & A \cap R \arrow[d, dash] & \\ & 1 & \end{tikzcd}$$

It is well known that a dense analytic subgroup $A$ of a connected Lie group $G$ is normal; in fact $A' = G'$ (see, e.g., [Hoch65, p. 190, Theorem 2.1]). Also, one knows (loc. cit.) that there is an abelian Lie subgroup $T$ such that $G = AT$. However, we need a sharper result which we prove now.

1.2. THEOREM. *Let $G$ be a connected Lie group, $A$ a dense analytic subgroup, and $T$ a maximal torus. Then $G = AT$. The maximal tori of $G$ are conjugate under the elements of $A$.*

*Proof.* The maximal tori in $G$ are conjugate. Hence if $S$ is second maximal torus and if we have $G = AT$, then there is an element $g = at$ with $a \in A$, $t \in T$, and $Tg^{-1} = S$. But then $aTa^{-1} = atTt^{-1}a^{-1} = gTg^{-1} = S$. The last assertion is therefore an easy consequence of $G = AT$. Moreover, if we can show $G = AT$ for *one* maximal torus $T$, then the result follows for all maxmal tori $T$.

(i) We first prove the result in the case where $G'$ is semisimple. Then $L(G)$ is reductive and the radical $R$ of $G$ is the identity component $Z_0$ of the center $Z$ of $G$. We claim that $AC(G) = G$, where $C(G)$ is the maximal compact subgroup of $R$; this will of course prove the claim in this special case. We shall factor $C(G)$ and operate in the factor group; hence we shall assume $C(G) = (1)$ and show $A = G$. By Lemma 1.1, the subgroup $G'$ is closed. From $L(G') = L(A') \subset L(A) \subset L(G') \oplus L(R)$ we find a vector subspace $V$ in $L(R)$ such that $L(A) = L(G') \oplus V$. But $R$ is a vector group, because $C(R) = (1)$. Hence $E = \exp V$ is a closed central subgroup which is contained in $A \cap R$ and which we shall factor. Subsequently, we assume $V = (0)$ without losing any generality. But now $L(G') = L(A)$ shows $G' = A$, and since $G'$ is closed, we have $G = \bar{A} = G' = A$. This finishes the assertion in the special case.

(ii) Now we prove the claim in the general case. It is clear that we are finished if we prove the assertion for the analytic group $AC(G)$ in place of $A$; hence we shall assume that $C(G) \subset A$. By Lemma 1.1 we know that $G'C(G)$ is a closed connected Lie subgroup $H$. We let $R_H$ denote the radical of $H$. Since $R_H$ is a characteristic subgroup of the normal subgroup $H$, we know that $R_H$ is a connected solvable normal subgroup of $G$ and is therefore contained in the radical $R$ of $G$. The factor group $H/R_H$ is semisimple, and $(G/R_H)' = G'R_H/R_H = G'C(G)/R_H$. Thus the factor group $G/R_H$ satisfies the hypotheses of the special case (i) above with $A/R_H$ replacing $A$. Hence if $K$ is a Lie subgroup of $G$ such that $R_H \subset K$ and $K/R_H$ is a maximal torus of $G/R_H$, we have $G/R_H = (A/R_H)(K/R_H)$, and thus $G = AK$. Now $K$ is a Lie group with a solvable normal closed subgroup $R_H$ such that $K/R_H$ is compact. But then $K = R_H T_1$ for some maximal torus $T_1$ (see, e.g., [Ho63, p. 46, 12.17] or [HM63, p. 31]). But this implies $G = AK = AR_H T_1 = AT_1$, and since $T_1$ is contained in some maximal torus $T$ of $G$, the theorem is proved. ▮

Since every maximal torus in a connected Lie group is contained in a compact subgroup and since all maximal compact subgroups are conjugate, the following is then an immediate consequence of Theorem 1.2:

1.3. COROLLARY. *Let $G$ be a connected Lie group, $A$ a dense analytic subgroup, and $K$ a maximal compact subgroup. Then $G = AK$, and all maximal compact subgroups are conjugate under the elements of $A$.*

In order to prove further consequences of Theorem 1.1 we first observe a lemma:

1.4. LEMMA. *Let $G$ be a connected Lie group and $T$ a maximal torus. Let $N$ be a closed connected normal subgroup containing the commutator subgroup $G'$. Let $R_N$ denote the radical of $N$ and assume that the center $Z(N/R_N)$ of $N/R_N$ is finite. Then $TN/N$ is the maximal torus of the abelian Lie group $G/N$.*

*Proof.* Let $C$ be that closed subgroup of $G$ containing $N$ for which $C/N$ is the maximal torus of $G/N$. If $R$ is the radical of $G$ and $S$ a Levi complement, then $S \subset G' \subset N$, whence $G = NR$ and $G/N = R/(N \cap R)$ as well as when $C/N = (C \cap R)/(N \cap R)$. Since $R_N$ is characteristic in $N$ we know that $R_N$ is a normal solvable connected subgroup of $G$, hence is contained in $R$. Thus $R_N \subset N \cap R$. Since $S$ is also a Levi complement for $R_N$ in $N$ we have $N = SR_N$. If $x \in N \cap R$, then $x = sr$ with $s \in S$ and $r \in R_N$. Then $s = xr^{-1} \in R \cap S \subset Z(S)$. Hence $N \cap R \subset R_N Z(S)$. Then $(N \cap R)/R_N \subset R_N Z(S)/R_N \subset Z(N/R_N)/R_N$, whence $(N \cap R)/R_N$ is finite and thus $(C \cap R)/R_N$ is compact. Now $C \cap R$ is a solvable Lie group with a connected closed normal subgroup $R_N$ such that $(C \cap R)/R_N$ is compact. Hence there is a compact subgroup $K$ of $C \cap R$ such that $C \cap R = KR_N$ (see, e.g., [Hoch65, p. 186]). Now let $T_R$ be a maximal compact subgroup of $R$ containing $K$. Then $T_R$ is a torus since $R$ is solvable, and $T_R(N \cap R)/(N \cap R) = T_R N/N$ is compact. But $C/N$ is the maximal torus of $G/N$, whence $T_R N/N \subset C/N \cap T_R N/N$. This shows $C = T_R N$. Now $T$ is a maximal torus of $G$. Hence there is a $g \in G$ such that $T_R \subset g^{-1}Tg$. Hence $C = gGg^{-1} = gT_R g^{-1} \cdot gNg^{-1} \subset TN$. But $TN/N$ is compact hence contained in $C/N$. Thus $C = TN$ which is what we had to show. ∎

1.5. THEOREM. *Let $G$ be a connected Lie group such that $Z((G')^-/\mathrm{Rad}(G')^-))$ is finite. Let $K$ be any compact connected subgroup containing a maximal torus of $G$. Let the analytic subgroup $A$ containing the closure of $G'$ be dense in $G$. Then the analytic subgroup $E$ generated by $L(A) \cap L(K)$ is dense in $K$ and has finite index in $A \cap K$.*

*Proof.* We set $N = (G')^-$ and apply Lemma 1.4 to see that the maximal torus of the connected abelian Lie group $G/N$ is $KN/N$. The dense analytic subgroup $A/N$ satisfies $(A/N)(KN/N) = G/N$, since $G = AK$ by Theorem 1.2. Thus $L(G/N) = L(A/N) + L(KN/N)$ and we can write $L(A/N) = V \oplus (L(A/N) \cap L(KN/N))$ with a vector space $V$ such that $L(G/N) = V \oplus L(KN/N)$. The image of $V$ in $G/N$ under the exponential function of $G/N$ is a closed vector subgroup $H/N$ contained in $A/N$. But then $(A/N) \cap (KN/N)$ is the projection of $A/N$ into $KN/N$ along $H/N$ and is, therefore, an analytic dense subgroup of $KN/N$. By the modular law we have $A \cap KN = (A \cap K)N$, and since $N$ is connected, $(A \cap K)N$ is a dense analytic subgroup of $KN$. The isomorphism $k(N \cap K) \to kN$: $K/(N \cap K) \to NK/N$ maps $(A \cap K)/(N \cap K)$ isomorphically onto $(A \cap K)N/(N \cap K)$. Hence $(A \cap K)/(N \cap K)$ is a dense analytic subgroup of the torus $K/(N \cap K)$. Now $N \cap K$ is a compact Lie group, and thus $(N \cap K)/(N \cap K)_0$ is finite. Now $(A \cap K)/(N \cap K)_0$ is dense in $K/(N \cap K)_0$ and has finitely many arc components. But then the arc component $((A \cap K)/(N \cap K)_0)_a$ of the identity is still dense in the torus $K/(N \cap K)_0$. Hence the arc component $(A \cap K)_a$ of the identity in $A \cap K$ is dense in $K$. But this arc component is the analytic subgroup $E$ generated by $L(A) \cap L(K)$. Morover, $E$ has finite index in $A \cap K$. ∎

The following is the most important consequence of the preceding result.

1.6. THEOREM. *Let $\pi: G \to Gl(n)$ be a continuous real or complex representation of a connected Lie group. Let $L(\pi): L(G) \to gl(n)$ be the induced representation on the Lie algebra level. Let $K$ be a compact subgroup of $(\pi(G))^-$ containing a maximal torus. Let $H$ be the arc component of the* *identity in $\pi^{-1}(K)$. Then $H$ is the analytic subgroup generated by $L(\pi)^{-1}L(K)$, and $(\pi(H))^{-}=K$.*

*Proof.* The subgroup $G^*=(\pi(G))^-$ of $Gl(n)$ is a connected linear Lie group. Hence its commutator subgroup $G^{*\prime}$ is closed (see [Hoch65, p. 224]). The subgroup $A=\pi(G)$ is a dense analytic subgroup of $G^*$ containing $G^{*\prime}$. Since $G^{*\prime}$ is linear, all of its semisimple factors have finite center (see [Hoch65, p. 221]). Hence $G^{*\prime}/\mathrm{Rad}(G^{*\prime})$ has a finite center. Thus Theorem 1.5 applies and shows that the analytic subgroup $E$ generated in $G^*$ by $L(A)\cap L(K)$, i.e., the arc component of 1 in $A\cap K$ is dense in $K$. Now $\pi^{-1}(K)=\pi^{-1}(A\cap K)$. Let $H$ be the identity component of this group. Then $H$ is analytic, and $L(H)$ consists of all $x\in L(G)$ with $\exp tx\in H$ for all $t$, i.e., with $\pi(\exp tx)\in K$ for all $t$, i.e., with $L(\pi)(x)\in L(K)$. Thus $L(H)=L(\pi)^{-1}(K)$. Since $L(\pi)(L(H))=L(A)\cap L(K)$ we know $\pi(H)=E$, and thus, in particular, $(\pi(H))^-=K$, since $E$ is dense in $K$. ∎

These results permit us to associate with any representation $\rho\colon L\to gl(n)$ of a Lie algebra a concept of "compactness for a subalgebra" of $L$.

1.7. Definition. Let $\rho\colon L\to gl(n)$ be a representation of a (finite dimensional) Lie algebra. For a subalgebra $M\leqslant L$ we set

(i) $M^\rho=(\langle e^{\rho(M)}\rangle)^-=$ closure in $Gl(n)$ of the analytic subgroup generated by all $e^{\rho(x)}$, $x\in M$.

For a subgroup $K$ of $Gl(n)$ we set

(ii) $K_\rho=\rho^{-1}(L(K_a))=\{x\in L\colon e^{t\rho(x)}\in K \text{ for all } t\in\mathbb{R}\}$ (where $K_a$ is the arc component of the identity in $K$, i.e., the largest analytic subgroup of $Gl(n)$ contained in $K$).

We shall say that a subalgebra $M$ of $L$ is $\rho$-compact if $M^\rho$ is compact.

The following remarks are almost immediate:

1.8. *Remarks.* Let $\rho\colon L\to gl(n)$ denote a representation. Then:

(i) The assignments $M\to M^\rho$ and $K\to K_\rho$ are monotone.

(ii) $\{0\}^\rho=\{1\}$, $\{1\}_\rho=\ker\rho$.

(iii) $M^\rho$ is a closed connected Lie subgroup of $Gl(n)$.

(vi) If $M\leqslant L$ is a subalgebra of $L$ and $K\leqslant Gl(n)$ is a closed subgroup of $Gl(n)$, then

$$K\geqslant M^\rho \qquad \text{if and only if } K_\rho\geqslant M.$$

(v) ${M^\rho}_\rho\geqslant M$ and $K\geqslant {K_\rho}^\rho$ for all subalgebras and all closed subgroups of $Gl(n)$, respectively.

In lattice theory, customarily, the assignments (i) between the lattices of subalgebras of $L$ and the closed subgroups of $Gl(n)$ are called a *Galois connection* if they satisfy condition (iv) (see, e.g., [GHKLMS80, pp. 18–29]).

1.9. *Remarks.* Under the same hypotheses as in 1.8 the following propositions hold:

(vi) $M + \ker \rho = M^{\rho}{}_{\rho}$ if the analytic subgroup $\langle e^{\rho(M)} \rangle$ generated in $Gl(n)$ is closed.

(vii) $K = K_{\rho}{}^{\rho}$ if and only if the analytic subgroup generated in $K$ by $L(K) \cap \rho(L)$ is dense in $K$.

*Proof.* (vi) We have $x \in M^{\rho}{}_{\rho}$ if and only if $e^{t\rho(x)} \in \langle e^{\rho(M)} \rangle^{-}$ for all $t$. This means $\rho(x) \in L(\langle e^{\rho(M)} \rangle^{-})$. If $\langle e^{\rho(M)} \rangle$ is closed, then $\rho(M) = L(\langle e^{\rho(M)} \rangle) = L(\langle e^{\rho(M)} \rangle^{-})$, and the assertion follows.

(vii) The analytic subgroup $\langle e^{\rho(K_{\rho})} \rangle$ is the one generated by $\rho(K_{\rho}) = \rho(\rho^{-1}(L(K)) = L(K) \cap \rho(L)$. Its closure is $K_{\rho}{}^{\rho}$, whence the assertion. ∎

The following remarks are deeper. First, it is clear that maximal $\rho$-compact subalgebras exist in $L$, since $L$ is finite dimensional. But the following results are no longer on the surface since they need the earlier theorems.

1.10. THEOREM. *Let $\rho: L \to gl(n)$ be a representation of a finite dimensional Lie algebra.*

(i) *If $K$ is a compact subgroup of $L^{\rho}$ containing a maximal torus then $K = K_{\rho}{}^{\rho}$. If, in addition, $K$ is a maximal compact subgroup of $L^{\rho}$, then $K_{\rho}$ is a maximal $\rho$-compact subalgebra of $L$.*

(ii) *If $M^{\rho}$ is a maximal $\rho$-compact subalgebra of $L$, then $M = M^{\rho}{}_{\rho}$* and *$M^{\rho}$ is a maximal compact subgroup of $L^{\rho}$. The analogous statement with "maximal" replaced by "maximal abelian" remains true.*

*Proof.* (i) Let $K$ be a maximal compact subgroup of $L^{\rho}$. Let $G$ denote the simply connected Lie group with $L = L(G)$. Then there is a unique representation $\pi: G \to Gl(n)$ with $L(\pi) = \rho$. Let $H$ be the analytic subgroup of $G$ generated by $K_{\rho} = L(\pi)^{-1}(K)$. We note that $\pi(H)$ is the analytic subgroup generated by $e^{L(\pi)(K_{\rho})} = e^{\rho(K)_{\rho}}$. Hence $K_{\rho}{}^{\rho} = (\pi(H))^{-}$. But by Theorem 1.6 we have $(\pi(H))^{-} = K$. Hence $K = K_{\rho}{}^{\rho}$. Now let $M$ be a $\rho$-compact subalgebra of $L$ with $K_{\rho} \leqslant M$. Then $K = K_{\rho}{}^{\rho} \leqslant M^{\rho}$ and $M^{\rho}$ is compact by definition. The maximality of $K$ then implies $K = M^{\rho}$. This implies, in particular, $M^{\rho} \leqslant K$ which is equivalent to $M \leqslant K_{\rho}$ by Remark 1.9 (iv). Since $K_{\rho}{}^{\rho} = K$ is compact, $K_{\rho}$ is a $\rho$-compact subalgebra of $L$, and then $M = K_{\rho}$ shows that in fact $K_{\rho}$ is a maximal $\rho$-compact subalgebra of $L$.

(ii) Let $M$ be a maximal $\rho$-compact subalgebra of $L$. Let $K$ be a maximal compact subgroup of $L^\rho$ with $M^\rho \leqslant K$. Then $M \leqslant K_\rho$ by Remark 1.8(iv). Since $K_\rho$ is $\rho$-compact because of $K_\rho{}^\rho \leqslant K$ by Remark 1.8(v), and since $M$ is maximal, we conclude $M = K_\rho$. Hence $M^\rho = K_\rho{}^\rho = K$ in view of (i) above. Hence $M^\rho$ is a maximal compact subgroup of $L$. Also $M \leqslant M^\rho{}_\rho$ by Remark 1.8(v). Further, $M^\rho{}_\rho{}^\rho = M^\rho$ (which follows from 1.8(v) as in the case of any Galois connection), whence $M^\rho{}_\rho$ is a $\rho$-compact subalgebra of $L$. The maximality of $M$ then shows $M = M^\rho{}_\rho$. The same proof works verbatim for the abelian case. ∎

For the next result, we have to recall the concept of the inner automorphism group of a (finite dimensional) Lie algebra $L$.

1.11. Definition. Let $L$ be a finite dimensional Lie algebra. Then $\operatorname{Aut} L$ denotes the group of all automorphisms of $L$, and $\operatorname{Inn} L$ denotes the subgroup of $\operatorname{Aut} L$ generated by all automorphisms of the form $e^{\operatorname{ad} x}$ with $x \in L$. We call $\operatorname{Inn} L$ the group of all *inner automorphisms*.

Recall that $\operatorname{Aut} L$ is a closed subgroup of $Gl(L)$ hence is a Lie group. Its Lie algebra is the Lie algebra $\operatorname{Der} L$ of all derivations of $L$. The group $\operatorname{Inn} L$ is the analytic subgroup generated in $\operatorname{Aut} L$ by the Lie algebra $\operatorname{ad} L$ of all inner derivations.

1.12. Theorem. *Let $\rho: L \to gl(n)$ denote a representation of the Lie algebra $L$. Then the maximal $\rho$-compact subalgebras of $L$ are conjugate under inner automorphisms of $L$. In particular, if $M$ is a maximal $\rho$-compact and $N$ any $\rho$-compact subalgebra of $L$, then there is an inner automorphism $\gamma \in \operatorname{Inn} L$ such that $\gamma(N) \subset M$.*

Before we prove this theorem, we establish a lemma

1.13. Lemma. *For a representation $\rho: L \to gl(n)$ of a Lie algebra $L$ the following propositions hold:*

(i) $\rho(e^{\operatorname{ad} x} y) = e^{\operatorname{ad} \rho(x)} \rho(y)$ *for all $x, y \in L$.*

(ii) $(e^{\operatorname{ad} x} M)^\rho = e^{\operatorname{ad} \rho(x)}(M^\rho)$ *for all $x \in L$ and each subalgebra $M$ of $L$.*

(iii) *If $\pi: G \to Gl(n)$ is any representation of a Lie group $G$ with $L(G) = L$ and $L(\pi) = \rho$, then*

$$(e^{\operatorname{ad} x} M)^\rho = \pi(\exp_G x) M^\rho \pi(\exp x)^{-1} \qquad \textit{for} \quad x \in L.$$

*Proof.* (i) We observe that $\rho(e^{\operatorname{ad} x} y) = \rho(\sum_{n=0}^{\infty} (1/n!)(\operatorname{ad} x)^n y) = \sum_{n=0}^{\infty} (1/n!)(\operatorname{ad} \rho(x))^n \rho(y) = e^{\operatorname{ad} \rho(x)} \rho(y)$.

(ii) $(e^{\operatorname{ad} x} M)^\rho = \langle e^{\rho(e^{\operatorname{ad} x} M)} \rangle^- = \langle e^{e^{\operatorname{ad} \rho(x)} \rho(M)} \rangle^-$ by (i) above. Now for two matrices $S$ and $T$ we always have $e^{\operatorname{ad} S} T = e^S T e^{-S}$. Hence

$e^{\operatorname{ad}\rho(x)}\rho(M)=e^{\rho(x)}\rho(M)e^{-\rho(x)}$. For each automorphism $I$ of a Banach algebra we have $I(e^X)=e^{I(X)}$. Hence $(e^{\operatorname{ad}x}M)^\rho=\langle e^{\rho(x)}e^{\rho(M)}e^{-\rho(x)}\rangle^-=e^{\rho(x)}\langle e^{\rho(M)}\rangle^-e^{-\rho(x)}=e^{\operatorname{ad}\rho(x)}M$, which establishes (ii).

(iii) The claim follows from (ii) and the fact that $\pi(\exp_G x)=e^{L(\pi)(x)}=e^{\rho(x)}$ holds for all $x\in L$. ∎

*Proof.* Now we are ready for the proof of Theorem 1.12. Let $G$ be the simply connected Lie group with $L=L(G)$ and let $\pi\colon G\to Gl(n)$ be the representation with $L(\pi)=\rho$. Let $M_j$, $j=1,2$, be two maximal $\rho$-compact subalgebras of $L$. Then $M_j^\rho$, $j=1,2$, are two maximal compact subgroups of $L^\rho$ by Theorem 1.10(ii). By Corollary 1.3, we have $L^\rho=\pi(G)M_1{}^\rho=\pi(G)M_2{}^\rho$, and $M_1$ and $M_2$ are conjugate under elements of $\pi(G)$. This means that there is a $g\in G$ such that $M_2{}^\rho=\pi(g)M_1{}^\rho\pi(g)^{-1}$. Now we find elements $x_1,...,x_n\in L(G)=L$ such that $g=\exp_G x_1\cdots\exp_G x_n$. We use Lemma 1.13(iii) and recursion to show $M_2{}^\rho=(e^{\operatorname{ad}x_1}\cdots e^{\operatorname{ad}x_n}M_1)^\rho$. Then Theorem 1.10(ii) applies to show $M_2=\gamma M_1$ with $\gamma=e^{\operatorname{ad}x_1}\cdots e^{\operatorname{ad}x_n}\in\operatorname{Inn}L$. The last assertion of 1.12 is then clear, and the proof of 1.12 is complete. ∎

1.14. DEFINITION. An element $x$ in a Hausdorff topological group $G$ is called *compact* if the subgroup $\langle x\rangle^-$ generated by $x$ in $G$ is compact. The set of all compact elements in $G$ will be denoted tor $G$.

In the following proof, for an analytic subgroup $S$ of a Lie group $G$, we write $S^\#=\exp L(S)$.

1.15. THEOREM. *Let $G$ be a Lie group with a dense analytic subgroup $A$ such that* $\exp L(A)\subset\operatorname{tor}G$. *Then $G$ is compact.*

*Proof.* We prove this claim by induction with respect to the dimension of $G$. So let $G$ be a counterexample to this theorem with minimal dimension; we shall derive a contradiction. A one-dimensional Lie group cannot be a counterexample to the theorem, and thus $\dim G>1$. If $A=G$, then $G^\#\subset\operatorname{tor}G$ and $G$ must be compact because of the existence of a manifold factor in noncompact Lie groups (see, e.g., [Hoch65, p. 180]). Hence, in a counterexample, $A\neq G$. Then $G'=A'\subset A$ (see [Hoch65, p. 190]), and so $G$ cannot be semisimple. We claim that $G$ cannot be solvable. If not, then $A$ is solvable, and we find an abelian characteristic analytic subgroup $I\neq\{1\}$ in $A$; in particular, $N=\bar{I}$ is normal in $G$. Now $I\subset A^\#\subset\operatorname{tor}G$, whence $I\subset N\cap\operatorname{tor}G=\operatorname{tor}N$. However, a connected abelian Lie group consisting of compact elements only is compact, because it cannot have a nondegenerate vector group factor. Now $AN$ is a dense analytic subgroup of $G$ contained in $N\cdot\operatorname{tor}G$. Hence $AN/N$ is a dense analytic subgroup of $G/N$ with $(AN/N)^\#\subset\operatorname{tor}G/N$. But if $N\neq\{1\}$, then $\dim G/N<\dim G$, and $G/N$

cannot be a counterexample. Hence $G/N$ is compact. But this implies that $G$ is compact since $N$ is compact. Thus $G$ is not a counterexample for the theorem. This contradiction shows $N=\{1\}$ and thus $I=\{1\}$, a contradiction. This proves the claim that $G$ cannot be solvable. Now let $R$ denote the radical of $A$. Since $A$ is dense in $G$ and $G$ is not solvable, $\bar{R}\neq G$. From $R^{\#}\subset A^{\#}\subset \operatorname{tor} G$ we again conclude $R^{\#}\subset \operatorname{tor}\bar{R}$. Since $G$ was a counterexample of minimal dimension, the group $\bar{R}$ cannot be a counterexample, hence $\bar{R}$ is compact. Considering $N=\bar{R}$ instead of $N=\bar{I}$ in the above argument we find now that $\bar{R}=\{1\}$, i.e., that $A$ is semisimple. But then $G'=A'=A$ by an earlier observation. Hence $A$ is a normal Levi subgroup of $G$ and $L(G)$ is reductive. Since $G$ is not compact, $A$ cannot be compact, and thus $L(A)$ contains a subalgebra isomorphic to $sl(2,\mathbb{R})$, and so it contains a two-dimensional nonabelian solvable subalgebra which generates an analytic subgroup $H$ with $H^{\#}\subset A^{\#}\subset \operatorname{tor} G$, whence $H^{\#}\subset \operatorname{tor}\bar{H}$. Since $\bar{H}\neq G$ we conclude again that $\bar{H}$ is compact. But any compact solvable connected Lie group is abelian and this is a contradiction, since $H$ is nonabelian. This final contradiction proves the theorem. ∎

We conclude this section with some observations on examples, which illustrate some of the complications we have encountered on the way to the principal results of this section.

1.16. EXAMPLE. Let $A$ be the group of all matrices

$$\begin{pmatrix} e^{is} & 0 & u \\ 0 & e^{\sqrt{2}is} & v \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $G$ be the closure of $A$ in $Gl(3,\mathbb{R})$. Then $A$ is a dense analytic subgroup of $G$. The Lie algebra of $A$ is the Lie subalgebra in $gl(n,\mathbb{R})$ of all matrices

$$(s,t,u,v)=\begin{pmatrix} is & 0 & u \\ 0 & it & v \\ 0 & 0 & 0 \end{pmatrix}$$

with $t=\sqrt{2}\,s$, while the Lie algebra of $G$ consists of all matrices $(s,t,u,v)$. If $\rho\colon L(A)\to gl(3)$ is the inclusion, then the set of matrices $(s,\sqrt{2}\,s,0,0)$ is $\rho$-compact in $L(A)$ and is maximal with respect to this property.

1.17. EXAMPLE. Let $S$ denote the universal covering group of $Sl(2,\mathbb{R})$ (see [HH85a]). Let $Z$ be its center which is generated by an element $z$ and is infinite cyclic. Let $E$ be a one parameter subgroup of $S$ passing through $Z$. Then $E\cong\mathbb{R}$. We consider the direct product $S\times\mathbb{R}$ and consider the discrete central subgroup $D$ of all $(z^m, m+\sqrt{2}\,n)$: $m, n \in \mathbb{Z}$ and set $G=(S\times\mathbb{R})/D$. Then $G$ is a four-dimensional reductive Lie group which is homeomorphic to $\mathbb{R}^2\times\mathbb{T}^2$. Its radical is $(Z\times\mathbb{R})/D\cong\mathbb{T}$, and its commutator subgroup $G'$ is the dense analytic subgroup $(S\times(\mathbb{Z}+\sqrt{2}\,\mathbb{Z}))/D$. A maximal compact subgroup $K$ is given by $(E\times\mathbb{R})/D$, and $G'\cap K$ is the dense analytic subgroup of $K$ given by $(E\times(\mathbb{Z}+\sqrt{2}\,\mathbb{Z}))/D$. The group $G$ has no faithful finite dimensional linear representation. The Lie algebra is $L(G)=sl(2,\mathbb{R})\oplus\mathbb{R}$.

A general construction and characterization of Lie groups with dense analytic subgroups is indicated in the following proposition, whose proof is largely computational and is left as an exercise for the reader:

1.18. Proposition. *Let $H$ be a connected Lie group with a closed subgroup $U$, let $C$ be a compact connected group, and let $f\colon U\to C$ be an injective morphism of topological groups with a dense image. Let $\alpha\colon C\to \operatorname{Aut} H$ denote a morphism of topological groups such that $\alpha$ and $f$ are linked by the following conditions*

(i) $\alpha(f(u))(h)=uhu^{-1}$ *for all* $u\in U$, $h\in H$.

(ii) $\alpha(C)(U)\subset U$ *and* $f(\alpha(c)(u))=c^{-1}f(u)c$ *for all* $u\in U$, $c\in C$.

*Let $H\times_\alpha C$ denote the semidirect product with multiplication $(h,c)(h',c')=(h\alpha(c)(h'),cc')$. Then $D=\{(u^{-1},f(u))\colon u\in U)$ is a closed normal subgroup, giving a Lie group $G=(H\times_\alpha C)/D$ and a quotient morphism $q\colon H\times_\alpha C\to G$. Then $A=q(H\times\{1\})$ is a dense analytic subgroup of $G$ and $K=q(\{1\}\times K)$ is a compact connected subgroup such that $G=AK$. Moreover, $A\cap K=(U\times f(U))/D\cong f(U)$. In particular, $A\cap K$ is connected if and only if $U$ is connected.*

*Conversely, if $G$ is a connected Lie group, $A$ a dense analytic subgroup, and $K$ a connected compact subgroup such that $G=AK$, then $G$, $A$, and $K$ arise in precisely this fashion with suitable Lie groups $H$, $U$, and morphisms $f\colon U\to C=K$, $\alpha\colon K\to \operatorname{Aut} H$ (where $H$ may be taken to be $A$ with its intrinsic Lie group topology).*

1.19. Example. In the construction of 1.18, we take $H=\mathbb{R}^2$, $U=\mathbb{Z}^2$, $C=\mathbb{T}$. We set $f\colon U\to C$, $f(m,n)=\pi m+\sqrt{2}\,n+\mathbb{Z}$. Then $G=\mathbb{T}^3$ and $A\cap K=f(U)=(\pi\cdot\mathbb{Z}+\sqrt{2}\cdot\mathbb{Z}+\mathbb{Z})/\mathbb{Z}$.

In this example $A\cap K$ is totally disconnected dense in $K$. It is instructive to compare this example with Theorem 1.5. There is one hypothesis for Theorem 1.5 which is not satisfied in Example 1.19, and that is the assumption that $K$ should contain a maximal torus of $G$.

## 2. $\rho$-Compact Elements in a Lie Algebra

2.1. Definition. Let $\rho: L \to gl(n)$ be a representation of a finite dimensional Lie algebra. An element $x \in L$ will be called *$\rho$-compact* if $(e^{\mathbb{R} \cdot \rho(x)})^{-}$ is compact in $Gl(n)$, i.e., if $\mathbb{R} \cdot x$ is a $\rho$-compact subalgebra of $L$ in the sense of Definition 1.8. The set of all $\rho$-compact elements of $L$ will be denoted $\operatorname{comp}_\rho L$.

We consider the following lemma as familiar.

2.2. Lemma. *Let $V$ be a finite dimensional real vector space and $X: V \to V$ an endomorphism of $V$. Let $G$ be the closure of $e^{\mathbb{R} \cdot X}$ in $Gl(V)$. Then the following conditions are equivalent*:

(1) *$G$ is compact.*

(2) *$X$ is semisimple and has purely imaginary spectrum.*

We now have the following characterization theorem for $\rho$-compact subalgebras of a Lie algebra $L$.

2.3. Theorem. *Let $M$ be a subalgebra of a finite dimensional Lie algebra $L$ and $\rho$: $L \to gl(V)$ a representation on a finite dimensional vector space $V$. Let $M_{\max}$ denote an arbitrary maximal $\rho$-compact subalgebra of $L$. Then the following conditions are equivalent*:

(1) *$M$ is $\rho$-compact.*

(2) $M \subset \operatorname{comp}_\rho L$.

(3) *For each $x \in M$, the endomorphism $\rho(x)$ of $V$ is semisimple and has purely imaginary spectrum.*

(4) *There is an inner automorphism $\gamma \in \operatorname{Inn} L$ of $L$ such that $\gamma(M) \subset M_{\max}$.*

(5) *There is a positive definite quadratic form $(\cdot \mid \cdot)$ on $V$ such that $(\rho(x)(u) \mid v) = -(u \mid \rho(x)(v))$ for all $x \in M$ and $u, v \in V$.*

*Proof.* The equivalence of (2) and (3) is an immediate consequence of Lemma 2.2. Condition (2) translates into the equivalent condition

(2′) $e^{\rho(M)} \subset \operatorname{tor} L^\rho$.

By Theorem 1.15, this condition is equivalent to

(1′) $M^\rho$ is compact.

But (1) and (1′) are obviously equivalent so that (1), (2), and (3) are all equivalent. The equivalence of (1) and (4) is a consequence of

Theorem 1.13. It remains to establish the equivalence of (5) with the other conditions. First we claim that (5) is equivalent to

(5′) $(g(u) \mid g(v)) = (u \mid v)$ for all $g \in M^\rho$ and $u, v \in V$.

Since $e^{\rho(M)}$ generates a dense analytic subgroup of $M^\rho$, condition (5′) is equivalent to

(5″) $(e^{\rho(x)}u \mid e^{\rho(x)}v) = (u \mid v)$ for all $x \in M$ and all $u, v \in V$.

We define $f: \mathbb{R} \to \mathbb{R}$ by $f(t) = (e^{t\rho(x)}u \mid e^{t\rho(x)}v)$. Then $f$ is differentiable and $f'(t) = (\rho(x)e^{t\rho(x)}u \mid e^{t\rho(x)}v) + (e^{t\rho(x)}u \mid \rho(x)e^{t\rho(x)}v)$. Now (5″) is equivalent to the constancy of $f$ for all $x \in M$ and $u, v \in V$. This means the vanishing of $f'(t)$. But the vanishing of $f'(t)$ for all $t$, $x$, $u$, and $v$ is equivalent to condition (5) and this proves the claim that (5) and (5′) are equivalent. But (5′) says that $M^\rho$ is a closed subgroup of some orthogonal group and that is equivalent to (1′). ∎

2.4. Corollary. *Let $\rho: L \to gl(V)$ be a representation of $L$ on a finite dimensional vector space $V$. If $M$ is a $\rho$-compact subalgebra, then $V$ is a semisimple $M$-module; i.e., every $\rho(M)$-invariant vector subspace $V_1$ has an invariant vector space complement $V_2$ such that $V = V_1 \oplus V_2$.*

*Proof.* It suffices to select a scalar product on $V$ according to Theorem 2.3.5 and to take for $V_2$ the orthogonal complement of $V_1$. ∎

For our later applications, the most important case of a representation of a finite dimensional Lie algebra $L$ is its adjoint representation ad: $L \to gl(L)$. This requires a definition.

2.5. Definition. Let $L$ be a finite dimensional Lie algebra. A subalgebra $M$ of $L$ is said to be *compactly embedded* or ad-*compact* if it is ad-compact for the adjoint representation ad: $L \to gl(L)$. An element $x \in L$ is called *compact* if it is ad-compact. The set of compact elements in $L$ is called comp $L$. The algebra $L$ is called *compact* if it is compactly embedded into itself.

Theorem 2.3 has the following corollary as a special case:

2.6. Corollary. *Let $M$ be a subalgebra of a finite dimensional Lie algebra $L$. Let $M_{\max}$ denote an arbitrary maximal compactly embedded subalgebra of $L$. Then the following conditions are equivalent:*

(1) *$M$ is compactly embedded.*

(2) $M \subset \operatorname{comp} L$.

(3) *For each $x \in M$, the endomorphism* ad $x$ *of the vector space $L$ is semisimple and has purely imaginary spectrum.*

(4) *There is an inner automorphism* $\gamma \in \operatorname{Inn} L$ *of* $L$ *such that* $\gamma(M) \subset M_{\max}$.

(5) *There is a positive definite quadratic form* $(\cdot \mid \cdot)$ *on* $L$ *such that* $([u, x] \mid v) = (u \mid [x, v])$ *for all* $x, u, v \in M$.

Note that every compactly embedded subalgebra of a Lie algebra is itself compact whence Corollary 2.4 has the following consequence:

2.7. COROLLARY. *If* $K$ *is a compactly embedded subalgebra of* $L$, *then there exists a vector subspace* $P$ *of* $L$ *such that* $L = K \oplus P$ *and* $[K, P] \subset P$; *i.e.,* $P$ *is a* $K$*-module complement for* $K$. ∎

We will now investigate the structure of the set $\operatorname{comp} L$ of all compact elements in a Lie algebra.

2.8. PROPOSITION. *Let* $L$ *be a finite dimensional Lie algebra. Then the set* $\operatorname{comp} L$ *of all compact elements is invariant under scalar multiplication and under all inner automorphisms. If* $M_{\max}$ *denotes a maximal compactly embedded subalgebra, then* $\operatorname{comp} L = (\operatorname{Inn} L) \cdot M_{\max}$.

*Proof.* We have $x \in \operatorname{comp} L$ if and only if $\mathbb{R} \cdot x$ is compactly embedded if and only if $\mathbb{R} \cdot x \subset \operatorname{comp} L$. Moreover, Corollary 2.6(4) shows that $\operatorname{comp} L \subset (\operatorname{Inn} L) \cdot M_{\max}$. The reverse inclusion is clear from the same result. Hence $\operatorname{comp} L = (\operatorname{Inn} L) \cdot M_{\max}$, and the invariance under inner automorphisms is a trivial consequence. ∎

In fact it is easy to see from the definitions that $\operatorname{comp} L$ is invariant under arbitrary automorphisms of $L$.

For our later purposes it is important for us to know when $\operatorname{comp} L$ has inner points. In this direction we shall prove the following theorem:

2.9. THEOREM. *Let* $L$ *be a finite dimensional Lie algebra. Then an element* $x \in L$ *is in the interior* $\operatorname{int}(\operatorname{comp} L)$ *if and only if the centralizer* $Z(x, L) = \ker(\operatorname{ad} x)$ *of* $x$ *in* $L$ *is contained in* $\operatorname{comp} L$.

The proof will progress through a sequence of lemmas, some of which may be of independent interest.

2.10. LEMMA. *Let* $K$ *be a compactly embedded subalgebra of* $L$, *and suppose that* $x \in K$. *Then* $(\operatorname{ad} x)^{-1}(K) = K + \ker(\operatorname{ad} x)$.

*Proof.* We write $L = K \oplus P$ according to 2.7. For $y \in L$ we write $y = k + p$ with $k \in K$ and $p \in P$. Now $y \in (\operatorname{ad} x)^{-1}(K)$ if and only if $[x, y] \in K$. This means $[x, p] = [x, y] + [x, k] \in K$, since $[x, k] \in [K, K] \subset K$. But $[x, p] \in [K, P] \subset P$ by 2.7, whence $[x, p] \in K$ if and only if

$[x, p] = 0$ because of $K \cap P = \{0\}$. Hence $y \in (\operatorname{ad} x)^{-1}(K)$ if and only if $p \in \ker(\operatorname{ad} x)$. Thus $(\operatorname{ad} x)^{-1}(K) = K \oplus (P \cap \ker(\operatorname{ad} x)) \subset K + \ker(\operatorname{ad} x)$. But $(\operatorname{ad} x)(K + \ker(\operatorname{ad} x)) \subseteq [x, K] \subset K$, whence $K + \ker(\operatorname{ad} x) \subset (\operatorname{ad} x)^{-1}(K)$. ▮

2.11. LEMMA. *Under the circumstances of Lemma* 2.10, *the linear map* $T: L \times K \to L$, $T(u, h) = h + [u, x]$ *is surjective if and only if* $\ker(\operatorname{ad} x) \subset K$.

*Proof.* Obviously, $K \subset \operatorname{im} T$. We let $p: L \to L/K$ denote the quotient map. Then $T$ is surjective if and only if $pT: T \to L/K$ is surjective. Now $(u, h) \in \ker pT$ if and only if $[u, x] \in K$, which by Lemma 2.10 is equivalent to $u \in (\operatorname{ad} x)^{-1}(K) = +\ker(\operatorname{ad} x)$. The surjectivity of $pT$ is equivalent to the relation $\dim(\operatorname{im} pT) = \dim L/K = \dim L - \dim K$. So $\dim(\operatorname{im} pT) = \dim(L \times K) - \dim(\ker pT) = \dim L + \dim K - (\dim(K + \ker(\operatorname{ad} x)) + \dim K) = \dim L - \dim(K + \ker(\operatorname{ad} x)$. Hence $pT$ is surjective if and only if $\dim(K + \ker(\operatorname{ad} x)) = \dim K$, i.e., if and only if $\ker(\operatorname{ad} x) \in K$. ▮

2.12. LEMMA. *Let $K$ be a compactly embedded subalgebra of $L$ and $x \in K$. Then the function $(u, v) \mapsto e^{\operatorname{ad} u} v: L \times K \to L$ is open at the point* $(0, x)$ *if* $\ker(\operatorname{ad} x) \subset K$.

*Proof.* If $f: L \times K \to L$ is defined by $f(u, v) = e^{\operatorname{ad} u} v$, then the differential $df_{(0,x)}: L \times K \to L$ at the point $(0, x)$ is given by $df_{(0,x)}(u, h) = h + [u, x]$. Thus $df_{(0,x)} = T$ with $T$ as in Lemma 2.10. Hence $df_{(0,k)}$ is surjective if and only if $\ker(\operatorname{ad} x) \subset K$ by Lemma 2.10. By the implicit function theorem, this condition implies the openness of $f$ at $(0, x)$. ▮

Now we are ready for the proof of Theorem 2.9.

First we show that $x \in \operatorname{int}(\operatorname{comp} L)$ whenever $\ker(\operatorname{ad} x)$ is compactly embedded. Here we apply Lemma 2.12 with $K = \ker(\operatorname{ad} x)$. Whenever $y \in \operatorname{comp} L$, then $e^{\operatorname{ad} u} y \in \operatorname{comp} L$ for all $u \in L$ by Theorem 2.6. By 2.12, $\operatorname{im} f$ is a neighborhood of $x$, and by what we just saw, this neighbourhood is contained in $\operatorname{comp} L$.

Second we show that $\ker(\operatorname{ad} x) \subset \operatorname{comp} L$ whenever $x \in \operatorname{int}(\operatorname{comp} L)$. Now the set $V = (\operatorname{comp} L) - x$ is a neighborhood of 0. If $y \in V \cap \ker(\operatorname{ad} x)$, then $y = c - x$ with $c \in \operatorname{comp} L$ and $[x, y] = 0$. Now $[x, c] = [x, x - y] = 0$. Hence $x$ and $c$ are two commuting compact elements, whence $y = c - x$ is compact (indeed $(\mathbb{R}y)^{\operatorname{ad}} \subset (\mathbb{R}c)^{\operatorname{ad}}(\mathbb{R}x)^{\operatorname{ad}}$, and $(\mathbb{R}c)^{\operatorname{ad}}$ as well as $(\mathbb{R}x)^{\operatorname{ad}}$ is compact). Thus $V \cap \ker(\operatorname{ad} x) \cap \operatorname{comp} L$. Hence $\mathbb{R} \cdot (V \cap \ker(\operatorname{ad} x)) \subset \operatorname{comp} L$ by 2.8. However, since $V$ is a neighborhood of 0 and $\ker(\operatorname{ad} x)$ is a vector space we have $\mathbb{R} \cdot (V \cap \ker(\operatorname{ad} x)) = \ker(\operatorname{ad} x)$. This proves the second claim. Since $\ker(\operatorname{ad} x)$ is clearly the centralizer $Z(x, L)$ of $x$ in $L$, Theorem 2.8 is now proved. ▮

## 3. Lie Algebras with Compactly Embedded Cartan Algebras

In this section we assume familiarity with the theory of Cartan subalgebras of an arbitrary Lie algebra; as a source of reference we refer to [Bou75].

We begin by recalling that any Cartan algebra $H$ of a finite dimensional real Lie algebra is obtained in the form of $L^0(x)$ for a regular element $x \in L$ (see [Bou75, p. 20, Corollaire 1; p . 23, Théorème 1]). Here $L^0(x)$ denotes the set of all $y \in L$ with $(\mathrm{ad}\, x)^n\, y = 0$ for sufficiently large $n$.

Our first result illuminates the significance of compactly embedded Cartan subalgebras in the context of our study of comp $L$:

3.1. Theorem. *Let $H$ be a Cartan subalgebra of a finite dimensional real Lie algebra $L$. Let $x \in H$ be any regular element of $L$ with $H = L^0(x)$. Then the following two statements are equivalent:*

(1) $x \in \mathrm{int}(\mathrm{comp}\, L)$.

(2) *$H$ is a compactly embedded subalgebra.*

*Moreover, every compactly embedded Cartan algebra of $L$ is abelian.*

*Proof.* The commutativity of a compactly embedded Cartan algebra follows immediately from its nilpotency and the following lemma:

3.2. Lemma. *Any compactly embedded solvable subalgebra of a Lie algebra is abelian.*

*Proof of Lemma* 3.2. If $S$ is a solvable compactly embedded subalgebra of $L$, then $S^{\mathrm{ad}}$ is a compact connected solvable subgroup of the Lie group $L^{\mathrm{ad}}$ and is, therefore, abelian. Hence $(S + Z(L))/Z(L)$ is abelian with the center $Z(L) = \ker(\mathrm{ad})$. Thus $S + Z(L)$ is a nilpotent subalgebra of $L$ in which $Z(L)$ has an $S$-invariant complement $W$ by 2.4, for $S$ is compactly embedded in $S + Z(L)$ by 2.6. Now $S + Z(L) = W \oplus Z(L)$ with $[S, W] \subset W$. Hence $[S + Z(L), W] \subset W$. Hence $W$ is an ideal with $W \cong (S + Z(L))/Z(L)$. Thus $W$ is abelian, and so is $S + Z(L)$; in particular, $S$ is abelian. ∎

Now we complete the proof of 3.1. First we show that (1) implies (2): Let $x \in \mathrm{int}(\mathrm{comp}\, L)$. Since $H$ is nilpotent, then $H^{\mathrm{ad}}$ is a nilpotent connected Lie group in which $(\mathbb{R} \cdot x)^{\mathrm{ad}}$ is a compact connected subgroup. But compact connected subgroups in nilpotent Lie groups are central. Hence $x$ is central in $H$ modulo $Z(L)$, i.e., $[x, H] \subset Z(L) \subset Z(H)$. By Corollary 2.6, $\mathrm{ad}\, x$ is semisimple, and so we have $H = W \oplus Z(H)$ with an $\mathrm{ad}\, x$-invariant vector subspace $W$. Now $[x, W] \subset [x, H] \cap W \subset Z(H) \cap W = \{0\}$, whence $x \in Z(H)$. In other words, $H \subset \ker(\mathrm{ad}\, x)$. But from Theorem 2.9 we know that $\ker(\operatorname{ad} x) \subset \operatorname{comp} L$, and so $H \subset \operatorname{comp} L$ which implies (2) by Corollary 2.6.

Next we show that (2) implies (1): By the first part of the proof we know that any compactly embedded Cartan algebra $H$ is abelian, whence $H = L^0(x) = \ker(\operatorname{ad} x)$. Thus $\ker(\operatorname{ad} x) = H \subset \operatorname{comp} L$ by (2). But then (1) holds by Theorem 2.9. ∎

Note that not all Cartan algebras of $L$ will generally be compactly embedded, but if one of them is, they will all be abelian, as a quick complexification argument shows.

We now know that compactly embedded Cartan algebras exist precisely when the set $\operatorname{comp} L$ of compact elements in $L$ has interior points. We shall make this more precise.

3.3. PROPOSITION. *For an element $x$ in a finite dimensional real Lie algebra $L$, the following conditions are equivalent*:

(1) $x \in \operatorname{int}(\operatorname{comp} L)$

(2) $\ker(\operatorname{ad} x) = \bigcup \{H\colon$ *$H$ is a compactly embedded Cartan subalgebra of $L$ containing $x$*$\}$.

*Proof.* (1) implies (2): Every element of $\mathbb{R} \cdot x$ is semisimple by Corollary 2.5. Thus we may apply [Bou75, VII, p . 16, Proposition 10] and conclude that the Cartan algebras $H$ of $\ker(\operatorname{ad} x)$ are exactly those Cartan subalgebras of $L$ which are contained in $\ker(\operatorname{ad} x)$. Since $x$ is central in $\ker(\operatorname{ad} x)$, every such Cartan algebra contains $x$, and since $\ker(\operatorname{ad} x)$ is compactly embedded by Theorem 3.1 and Corollary 2.6, every such Cartan algebra is also compactly embedded. On the other hand, every compactly embedded Cartan subalgebra of $L$ is abelian by Theorem 3.1. Thus, if it also contains $x$, it must be contained in $\ker(\operatorname{ad} x)$. Thus the left hand side of (2) contains the right hand side. But $\ker(\operatorname{ad} x)$ is itself compactly embedded, and so it is itself a compact Lie algebra. As such it is the union of all of its Cartan subalgebras. After the preceding, every one of these is a Cartan subalgebra of $L$ which is compactly embedded and contains $x$. Hence the left hand side of (2) is contained in the right hand side.

(2) implies (1): Each compactly embedded Cartan subalgebra of $L$ is contained in $\operatorname{comp} L$. Hence the right hand side of (2) is contained in $\operatorname{comp} L$, and thus $\ker(\operatorname{ad} x) \subset \operatorname{comp} L$. By Theorem 3.1, this is equivalent to (1). ∎

We recall, of course, that $\ker(\operatorname{ad} x)$ is the centralizer $Z(x, L)$ of $x$ in $L$.

3.4. COROLLARY. *In any finite dimensional real Lie algebra L we have*

$$\operatorname{int}(\operatorname{comp} L) \subset \bigcup \{H\colon H \textit{ is compactly embedded Cartan algebra with } x \in H\}$$
$$\subset \operatorname{comp} L,$$

*and the regular elements contained in the middle set are all in* $\operatorname{int}(\operatorname{comp} L)$. *In particular,* $\operatorname{int}(\operatorname{comp} L)$ *is dense in the middle set.*

*Proof.* The containments are clear from the preceding. If $x$ is regular and $H = L^0(x)$ is compactly generated, then $H = L^0(x) = \ker(\operatorname{ad} x)$ and $H \subset \operatorname{comp} L$, whence $x \in \operatorname{int}(\operatorname{comp} L)$ by Theorem 3.1. ∎

3.5. PROPOSITION. *Any two compactly embedded Cartan subalgebras are conjugate under inner automorphisms.*

*Proof.* If $H_j$, $j = 1, 2$, are two compactly embedded Cartan algebras, then we find two maximal compactly embedded subalgebras $K_j$, $j = 1, 2$, with $H_j \subset K_j$. By Corollary 2.6 there is an inner automorphism $\gamma$ such that $\gamma(K_1) = K_2$. Hence $\gamma(H_1)$ and $H_2$ are two Cartan subalgebras of the compact Lie algebra $K_2$. Hence there is an element $x \in K_2$ such that $e^{\operatorname{ad} x}\gamma(H_1) = H_2$ (see, e.g., [Bou82, Sect. 2, Théorème 1 and Corollaires of Théorème 2]). ∎

3.6. DEFINITION. Let $M$ be a subalgebra of a Lie algebra $L$. We set

$$\text{(i)} \quad Z(M, L)^* = \{\alpha \in L^{\mathrm{ad}}\colon \alpha\beta = \beta\alpha \text{ for all } \beta \in M^{\mathrm{ad}}\}$$
$$= \text{centralizer of } M^{\mathrm{ad}} \text{ in } L^{\mathrm{ad}},$$
$$\text{(ii)} \quad N(M, L)^* = \{\alpha \in L^{\mathrm{ad}}\colon \alpha M^{\mathrm{ad}}\alpha^{-1} = M^{\mathrm{ad}}\}$$
$$= \text{normalizer of } M^{\mathrm{ad}} \text{ in } L^{\mathrm{ad}}.$$

By a slight abuse of language we shall call $Z(M, L)^*$ the centralizer and $N(M, L)^*$ the normalizer of $M$.

3.7. LEMMA. *If M is a compactly embedded abelian subalgebra of L, then* $N(M, L)^*/Z(M, L^*)$ *is finite.*

*Proof.* The group $M^{\mathrm{ad}}$ is compact abelian acting linearly on $L$, hence on the complexification $L_{\mathbb{C}} = \mathbb{C} \otimes L$. There is a finite set $R$ of characters $\chi$ of $M^{\mathrm{ad}}$ and a decomposition of $L_{\mathbb{C}}$ into a direct sum of isotypic $M^{\mathrm{ad}}$-submodules $V_\chi = \{v \in L\colon \alpha(v) = \chi(\alpha) \cdot v \text{ for all } \alpha \in M^{\mathrm{ad}}\}$. If $v \in N(M, L)^*$ and $\chi$ is an arbitrary character of $M^{\mathrm{ad}}$, then $(\chi \cdot v)(\alpha) = \chi(v\alpha v^{-1})$ defines a new character, and $(\chi, v) \to \chi \cdot v$ defines an action of $N(M, L)^*$ on the right on the character group of $M^{\mathrm{ad}}$. This action leaves the set $R$ invariant, hence defines a homomorphism of $N(M, L)^*$ into the finite group of all permutations of $R$. An element $v \in N(M, L)^*$ is in the kernel of this representation if and only if $\chi \cdot v = \chi$ for all $\chi \in R$. This means that for all $v \in V$ and all $\chi \in R$ we have $v\alpha v^{-1}(v) = \chi(v\alpha v^{-1}) \cdot v = \chi(\alpha) \cdot v = \alpha(v)$. It follows that $v\alpha v^{-1} = \alpha$ for all $\alpha \in M^{\mathrm{ad}}$, i.e., that $v \in Z(M, L)^*$. This proves the claim. ∎

3.8. Lemma. *If $H$ is a compactly embedded Cartan algebra of $L$, then $L(H^{\mathrm{ad}})$ is a Cartan algebra of $L(L^{\mathrm{ad}})$.*

*Proof.* Since $L(H^{\mathrm{ad}})$ is abelian, we have to show that $L(H^{\mathrm{ad}})$ is its own normalizer. Thus let $X$ be an element of $L(L^{\mathrm{ad}})$ such that $[X, L(H^{\mathrm{ad}})] \subset L(H^{\mathrm{ad}})$. This is tantamount to saying that $(\exp tX) H^{\mathrm{ad}} (\exp tX)^{-1} \subset H^{\mathrm{ad}}$, and this, due to the density of $e^{\mathrm{ad}\, H}$ in $H^{\mathrm{ad}}$, is equivalent to $(\exp tX)$ $e^{\mathrm{ad}\, H} (\exp tX)^{-1} \subset H^{\mathrm{ad}}$ for all $t \in \mathbb{R}$. This again means

$$[X, \mathrm{ad}\, H] \subset L(H^{\mathrm{ad}}).$$

By 1.10(ii), $H^{\mathrm{ad}}$ is a maximal torus of $L^{\mathrm{ad}}$. Hence Theorem 1.2 applies to show that $L^{\mathrm{ad}} = \langle e^{\mathrm{ad}\, L} \rangle H^{\mathrm{ad}}$. Translated to the Lie algebra level this means $L(L^{\mathrm{ad}}) = \mathrm{ad}\, L + L(H^{\mathrm{ad}})$. Thus we may write $X = \mathrm{ad}\, x + Y$ with suitable elements $x \in H$ and $Y \in L(H^{\mathrm{ad}})$. Since $\mathrm{ad}\, H \subset L(H^{\mathrm{ad}})$ and $H^{\mathrm{ad}}$ is abelian, we conclude $[Y, \mathrm{ad}\, H] = \{0\}$. Thus we obtain $\mathrm{ad}[x, H] = [\mathrm{ad}\, x, \mathrm{ad}\, H] \subset L(H^{\mathrm{ad}})$. In particular $\mathrm{ad}[x, H]$ and $\mathrm{ad}\, H$ are both subsets of $L(H^{\mathrm{ad}})$, hence commute, whence $\mathrm{ad}[[x, H], H] = [\mathrm{ad}[x, H], \mathrm{ad}\, H] = \{0\}$, and thus $[[x, H], H] \subset \ker \mathrm{ad} = Z(L) \subset H$, since $H$ is a Cartan algebra. Thus $[x, H]$ is in the normalizer of $H$, which is $H$, since $H$ is a Cartan algebra, and thus $[x, H] \subset H$. But then $x \in H$, since $H$ is its own normalizer. Hence $X \in \mathrm{ad}\, H + L(H^{\mathrm{ad}}) = L(H^{\mathrm{ad}})$ which we had to show. ∎

3.9. Proposition. *Let $T$ be a torus subgroup of a connected Lie group $G$ such that $L(T)$ is a Cartan algebra of $L(G)$. Then $Z(T, G) = T$ (where $Z(T, G)$ is the centralizer of $T$ in $G$).*

*Proof.* We prove this in several steps.

*Step* 1. The assertion is true if $G$ is semisimple: This is a consequence of [Wa72, Proposition 1.4.1.4, p. 110], since $L(T)$ is fundamental in view of [Wa72, Theorem 1.14 and Proposition 1.3.3.4, p. 100].

*Step* 2. The assertion is true if $G$ is solvable. It is convenient to prepare the following simple

Lemma. *Let $G$ be a group with a group $A$ acting as a group of automorphisms on $G$. Suppose that $H, N$ are subgroups of $G$ such that $N$ is* *normal with* $A \cdot N \subset N$ *and that the following conditions are satisfied*: (i) $\mathrm{Fix}(A, N) = \{1\}$. (ii) $\mathrm{Fix}(A, G/N) \subset HN/N$. (iii) $H \subset \mathrm{Fix}(A, G)$. *Then* $H = \mathrm{Fix}(A, G)$.

*Proof of the lemma.* Let $f \in \mathrm{Fix}(A, G)$. Then $fN \in \mathrm{Fix}(A, G/N)$, so by (ii) we have $f = hn$ with $h \in H$ and $n \in N$. Then $hn = f = a \cdot f = (a \cdot h)(a \cdot n)$ for all $a \in A$ in view of (iii). Thus $n = a \cdot n$ for all $a \in A$, whence $n = 1$ by (i), and so $f = h \in H$. ▌

Now we proceed by induction with respect to the dimension of $G$. We let $M$ be a minimal closed connected nondegenerate normal subgroup of $G$. Then $M$ is abelian and $\dim L(M) = 1$ or 2, since $L(M)$ is an irreducible $G$-module. If $T$ acts trivially on $L(M)$, then $L(M)$ is in $Z(L(T), L(G))$; but this centralizer is $L(T)$ since $L(T)$ is a Cartan algebra of $L(G)$. Hence $L(M) \subset L(T)$ and thus $M \subset T$. Now we apply the induction hypothesis to $G/M$ and find $Z(T/M, G/M) = T/M$, which implies $Z(T, G) \subset T$ and thus $Z(T, G) = T$. Therefore we now assume that $T$ does not act trivially on $L(M)$. Then the dimension of $L(M)$ cannot be 1 since $T$ is compact and connected. Thus $\dim L(M) = 2$, and then $T$ acts irreducibly. The fixed point set of $T$ is a vector subspace, hence by irreducibility must be singleton. Now we apply the lemma with $A$ as the group $T$ acting on $G$ under inner automorphisms and with $N = M$ and $H = T$. Condition (i) was just shown; condition (ii) holds by induction hypothesis, and condition (iii) is the trivial inclusion $T \subset Z(T), G)$. Hence the lemma applies and shows $T = Z(T, G)$. The induction is complete and Step 2 follows.

*Step* 3. Now let $R$ be the radical. Then $A = RT$ is a solvable subgroup, and $L(T)$ is still a Cartan subalgebra of $L(A)$. By Step 2 we therefore know that $Z(T, RT) = T$. By Step 1 we have $\mathrm{Fix}(T, G/R) = TR/R$ (with $T$ acting on $G$ and then on $G/R$ under inner automorphisms). Now we apply the lemma with $A = T$ acting under inner automorphisms, $N = R$, $H = T$, and observe that (i), (ii), (iii) are satisfied. Then the lemma shows $\mathrm{Fix}(T, G) = T$ and this is the assertion. ▌

3.10. COROLLARY. *If $H$ is a compactly embedded Cartan algebra of $L$, then* $Z(H, L)^* = H^{\mathrm{ad}}$.

*Proof.* Since $L(H^{\mathrm{ad}})$ is a Cartan algebra of $L(L^{\mathrm{ad}})$ by Lemma 3.8, Proposition 3.9 applies and establishes the claim. ▌

3.11. THEOREM. *Let $H$ be a compactly embedded Cartan algebra in a finite dimensional Lie algeba $L$. Then $N(H, L)^*/H^{\mathrm{ad}}$ is a finite group $W(H, L)$ acting on $H$ as follows: If $w = vH^{\mathrm{ad}}$ with $v \in N(H, L)^*$, then* $w \cdot x = v(x)$, *and* $e^{\mathrm{ad}(w \cdot x)} = v e^{\mathrm{ad}\, x} v^{-1}$.

*Proof.* Corollary 3.10 and Lemma 3.7 show that the goup $W(H, L)$ is finite. The dense analytic subgroup $e^{\operatorname{ad} L}$ of $L^{\mathrm{ad}}$ intersects the maximal torus $H^{\mathrm{ad}}$ in a dense subgroup whose identity component is the analytic subgroup generated by $\operatorname{ad} H$. This subgroup is invariant under inner automorphisms by elements of $N(H, L)^*$, since $e^{\operatorname{ad} L}$ is normal in $L^{\mathrm{ad}}$. Furthermore, we have $ve^{\operatorname{ad} x}v^{-1}(y) = \sum_{n=0}^{\infty} (1/n!)\, v(\operatorname{ad} x)^n v^{-1}(y) = \sum_{n=0}^{\infty} (1/n!)(\operatorname{ad} v(x))^n(y) = e^{\operatorname{ad} v(x)}(y)$. According to Theorem 1.10(ii) we know that $H = H^{\mathrm{ad}}{}_{\mathrm{ad}}$, whence $v(H) = H$. Thus the well defined action $(w, x) \to w \cdot x = v(x)$: $W(H, L) \times H \to H$ satisfies $e^{\operatorname{ad}(w \cdot x)} = ve^{\operatorname{ad} x}v^{-1}$. ∎

3.12. Definition. The group $W(H, L)$ is called the *Weyl group* of the compactly embedded Cartan algebra $H$ in $L$.

This machinery now allows us to conclude this section with the following important result:

3.13. Theorem. *Let $H$ be a compactly embedded Cartan algebra in a finite dimensional real Lie algebra $L$. Then there is a unique maximal compactly embedded subalgebra $K(H)$ such that $H \subset K(H)$. Moreover, $N(H, L)^* \subset K(H)^{\mathrm{ad}}$, and if $\gamma$ is any inner automorphism of $L$, then $K(\gamma(H)) = \gamma(K(H))$.*

*Proof.* Let $K$ be any maximal compactly embedded subalgebra of $L$ containing $H$. By Theorem 3.11, $N(H, L)^*$ is a compact subgroup of $L^{\mathrm{ad}}$, hence is contained in a maximal compact subgroup $C$ of $L^{\mathrm{ad}}$. Then $C_{\mathrm{ad}}$ is a maximal compactly embedded subalgebra of $L$ containing $H$ by Theorem 1.10(i). Hence by Theorem 1.12, there is an inner automorphism $\gamma$ of $L$ such that $C_{\mathrm{ad}} = \gamma(K)$. Thus $\gamma(H)$ and $H$ are Cartan subalgebras of $C_{\mathrm{ad}}$. Hence by Proposition 3.5 and the surjectivity of the exponential function on compact Lie groups there is an $x \in C_{\mathrm{ad}}$ such that with $\kappa = e^{\operatorname{ad} x}$ we have $\kappa\gamma(H) = H$. Then obviously $\kappa\gamma \in N(H, L)^* \subset C$. Hence $((\kappa\gamma) C (\kappa\gamma)^{-1})_{\mathrm{ad}} = C_{\mathrm{ad}}$. But now we claim that $((\kappa\gamma) C (\kappa\gamma)^{-1})_{\mathrm{ad}} = (\kappa\gamma)(C_{\mathrm{ad}})$. For this purpose we observe the following lemma.

3.14. Lemma. *If $v$ is an inner automorphism of a Lie algebra $L$ and if $C$ is a closed subgroup of $L^{\mathrm{ad}}$, then $(vCv^{-1})_{\mathrm{ad}} = v(C_{\mathrm{ad}})$.*

*Proof.* We have $x \in C_{\mathrm{ad}}$ if and only if

$$e^{t \operatorname{ad} x} \in C \qquad \text{for all } t \in \mathbb{R}.$$

But $v(e^{t \operatorname{ad} x})v^{-1} = e^{t \operatorname{ad} v(x)}$ as we have seen in the proof of 3.11. The lemma follows. ∎

But now $\kappa \in C$, whence by Lemma 3.14, we have $\kappa(C_{\mathrm{ad}}) = C_{\mathrm{ad}}$. As a consequence, we have $\gamma(C_{\mathrm{ad}}) = C_{\mathrm{ad}}$. Hence $K = C_{\mathrm{ad}}$, and this shows that $C_{\mathrm{ad}}$ is the only maximal compactly embedded subalgebra containing $H$.

Finally, if $\gamma$ is an inner automorphism of $L$, then $\gamma(K(H))$ is a maximal compactly embedded subalgebra of $L$ containing $\gamma(H)$, hence by uniqueness must agree with $K(\gamma(H))$. This completes the proof of the theorem. ∎

3.15. COROLLARY. *The Weyl group $W(H, L)$ of a compactly embedded Cartan algebra $H$ in a Lie algebra $L$ is the classical Weyl group of the torus $H^{\mathrm{ad}}$ in the compact group $K(H)^{\mathrm{ad}}$.*

*Proof.* By Theorem 1.10(i), $K(H)^{\mathrm{ad}}$ is a maximal compact group containing $H^{\mathrm{ad}}$. By the proof of 3.13, $N(H, L)^* \subset K(H)^{\mathrm{ad}}$, and so the normalizer of $H^{\mathrm{ad}}$ in $K(H)^{\mathrm{ad}}$ contains $N(H, L)^*$, which implies the corollary. ∎

3.16. PROPOSITION. *Let $L$ be a finite dimensional Lie algebra with a compactly embedded Cartan subalgebra $H$. Then $L$ decomposes into a direct sum $L = K(H) \oplus P(H)$ with a $K(H)$-submodule $P(H)$. In specific terms, $[K(L), P(L)] \subset P(L)$.*

*Proof.* This propostion is simply a special case of Corollary 2.7. ∎

In general, one cannot say very much about $[P(L), P(L)]$. A straightforward calculation, however, shows the following observation:

3.17. *Remark.* Under the circumstances of Proposition 3.15, the set $\{p \in P(H) \colon [p, P(H)] \subset P(H)\}$ is a subalgebra and a $K(H)$-submodule.

## 4. INVARIANT CONES AND COMPACTLY EMBEDDED LIE ALGEBRAS

A subset $W$ of a finite dimensional vector space $E$ will be called a *cone*, if $\bar{W} = W$, further $W + W \subset W$, and finally $\mathbb{R}^+ \cdot W \subset W$ (with the set $\mathbb{R}^+$ of nonnegative real numbers). We shall call the largest vector subspace $W \cap -W$ contained in $W$ the *edge* of the cone and denote it with $H(W)$. We shall call a cone *pointed* if its edge is singleton. The cone has inner points if and only if $W - W = E$. Such cones will be called *generating*. The following lemma is probably well known.

4.1. LEMMA. *Let $W$ be a pointed generating cone in a finite dimensional vector space $E$ and consider an interior point $w$ of $W$. If $C$ denotes the group of all automorphisms of $E$ leaving $W$ invariant and fixing $w$, then $C$ is compact.*

*Proof.* We consider the dual $E^\wedge$ of $E$ and the dual cone $W^* = \{f \in E^\wedge : f(w) \geqslant 0 \text{ for all } w \in W\}$. The group $C^\wedge$ of all adjoints $\hat{\gamma}$ of elements $\gamma \in C$ leaves $W^*$ invariant, since $C$ leaves $W$ invariant. Let $P = \{f \in E^\wedge : f(w) = 0\}$ denote the polar of $w$. Then $P \cap W^* = \{0\}$ since $w$ is an inner point of $W$; and since $w$ is fixed under $C$, the hyperplane $P$ is invariant under $C^\wedge$. The action of $C$ on $\mathbb{R} \cdot w$ is trivial, and then so is the action of $C^\wedge$ on $E^\wedge/P$, the dual module of $\mathbb{R} \cdot w$. Hence in fact all cosets $f + P$, $f \in E^\wedge$ are invariant under $C^\wedge$. Since $W$ is pointed cone, $W^*$ has an inner point $f$, and since $W$ is generating, $W^*$ is pointed. Thus, because of $P \cap W^* = \{0\}$, the set $K = W^* \cap (f + P)$ is a compact convex $C^\wedge$-invariant set. Consequently, the closed convex hull $K_0$ of $K \cup -K$ is likewise $C^\wedge$-invariant. But $K_0$ is a neighborhood of 0 in $E^\wedge$: Indeed $W^* \cap K_0$ is the closed convex hull of $\{0\} \cup K = [0, 1] \cdot K$, and $g = \frac{1}{2} \cdot f$ in the interior of this set, hence in the interior of $K_0$. But then $0 = \frac{1}{2}(g + (-g))$ is in the interior of $K_0$, as was asserted. Now $C^\wedge$ is a closed subgroup of $Gl(E^\wedge)$ leaving a convex compact neighborhood of 0 invariant. Hence $C^\wedge$ is compact. But $C$ is antiisomorphic to $C^\wedge$, hence is compact, too. ▮

4.2. Corollary. *Let $W$ be a pointed generating cone in a finite dimensional vector space $E$, and let $\rho: L \to gl(E)$ be a representation of the Lie algebra $L$. If an element $x \in L$ satisfies $e^{\mathbb{R} \cdot \rho(x)} W = W$ and $\rho(x)(w) = 0$ for some $w \in \operatorname{int}(W)$, then $x \in \operatorname{comp}_\rho L$. In particular, if $E = L$ and $\rho = \operatorname{ad}$, and if $x = w$, then $x \in \operatorname{comp} L$.*

*Proof.* The group $(\mathbb{R} \cdot x)^\rho$ is compact by Lemma 4.1, whence the assertion. ▮

4.3. Definition. A cone $W$ is in a finite dimensional Lie algebra $L$ is called invariant, if $e^{\operatorname{ad} x} W = W$ for all $x \in L$. This is clearly equivalent to the following statement:

$$L^{\operatorname{ad}} \cdot W = W.$$

It is a straightforward observation that for an invariant cone $W$ both the edge $H(W)$ and the vector subspace $W - W$ generated by $W$ are ideals of $L$. To assume that an invariant cone be both pointed and generating is not very much of a restriction, since the image of $W$ in $(W - W)/H(W)$ is a pointed invariant cone.

The following corollary allows us to connect the theory of invariant cones with the developments of the previous sections.

4.4. Corollary. *If $W$ is an invariant pointed cone in a finite dimensional Lie algebra, then $\operatorname{int}(W) \subset \operatorname{comp} L$. In particular, $\operatorname{comp} L$ has nonempty interior.*

*Proof.* This is an immediate consequence of Corollary 4.2 and Definition 4.3. ▮

4.5. Corollary. *If $W$ is a pointed invariant cone in a finite dimensional Lie algebra $L$, the following conclusions hold:*

(i) $\operatorname{int}(W) \subset \bigcup \{H\colon H$ *is a compactly embedded algebra of* $L\}$

(ii) *If $H$ is any compactly embedded Cartan algebra of $L$, then* $H \cap \operatorname{int}(W) \neq \emptyset$.

*In particular, all Cartan algebras of $L$ are abelian.*

*Proof.* (i) By Corollary 4.4 we have $\operatorname{int}(W) \subset \operatorname{comp} L$. The assertion (i) then follows from Corollary 3.4. In particular, there are compactly embedded Cartan algebras in $L$, and thus all Cartan algebras of $L$ are abelian.

(ii) Let $H$ be a compactly embedded Cartan algebra. By (i) above we find a compactly embedded Cartan algebra $H_1$ meeting the interior of $W$. Next we apply Proposition 3.5 in order to find an inner automorphism $\gamma$ of $L$ with $\gamma(H) = H_1$, whence $\emptyset \neq H_1 \cap \operatorname{int}(W) = \gamma(W) \cap \gamma(\operatorname{int}(W))$ (by the invariance of $W$!) $= \gamma(W \cap \operatorname{int}(W))$. This proves (ii). ▮

This information now allows us to prove a uniqueness theorem for invariant pointed cones in Lie algebras. We shall see that such cones are uniquely determined by their intersections with a fixed (but arbitrary) compactly embedded Cartan algebra.

4.6. Theorem (Uniqueness theorem for invariant cones). *Let $W$ be an invariant pointed generating cone in a finite dimensional real Lie algebra $L$. Let $H$ be any compactly embedded Cartan algebra (which exists by Corollary 4.5). Then $\operatorname{int}_L(W) = \operatorname{Inn}(L) \cdot \operatorname{int}_H(H \cap W)$. In particular, if $H_1$ and $H_2$ are compactly embedded Cartan algebras and $W_1$ and $W_2$ are invariant pointed generating cones of $L$ such that $H_1 \cap W_1$ is conjugate to $H_2 \cap W_2$ under inner automorphisms, then $W_1 = W_2$.*

*Proof.* First we observe that $\operatorname{int}_H(H \cap W) = H \cap \operatorname{int}(W)$: clearly the left side contains the right side; now let $w \in H \cap \operatorname{int}(W)$ and $h \in H \cap W$. Then $t \cdot h + (1-t) \cdot w \in H \cap \operatorname{int}(W)$ for $0 \leqslant t < 1$, and since all points of $H \cap W$ are of the form $t \cdot h + (1-t) \cdot w$ with a fixed $w \in H \cap \operatorname{int}(W)$ and suitable $h \in H$, $0 \in t < 1$, the claim follows. By Corollary 4.5(i) we have $\operatorname{int}(W) = \bigcup \{H_1 \cap \operatorname{int}(W)\colon H_1$ is a compactly embedded Cartan algebra of $L\}$. Hence we find an inner automorphism $\gamma$ for each compactly embedded Cartan algebra $H_1$ such that $H_1 \cap \operatorname{int}(W) = \gamma(H) \cap (\operatorname{int}(W)) = \gamma(H \cap \operatorname{int}(W))$ (by the invariance of $W$!) $= \gamma(\operatorname{int}_H(H \cap W))$ by what we saw in the beginning of the proof. We have now shown $\operatorname{int}_L(W) = \operatorname{Inn}(L) \cdot \operatorname{int}_H(H \cap W)$. Now suppose that $W_1$, $W_2$, $H_1$, and $H_2$ satisfy the hyoptheses in the theorem. By the preceding we then have $\operatorname{int}_L(W_1) = \operatorname{Inn}(L) \cdot \operatorname{int}_{H_1}(H_1 \cap W_1)$ and $\operatorname{int}_L(W_2) = \operatorname{Inn}(L) \cdot \operatorname{int}_{H_2}(H_2 \cap W_1)$. Since $H_1 \cap W_1$ is conjugate to $H_2 \cap W_2$ we know that $\operatorname{int}_{H_1}(H_1 \cap W_1)$ is conjugate to $\operatorname{int}_{H_2}(H_2 \cap W_2)$ under inner automorphisms, and thus $\operatorname{int}(W_1) = \operatorname{int}(W_2)$, whence $W_1 = W_2$. ▮

We return briefly to the general situation of a compactly embedded Cartan subalgebra $H$ of a finite dimensional Lie algebra $L$. We recall that $T = H^{\mathrm{ad}}$ is a torus subgroup of $L^{\mathrm{ad}} \subset gl(L)$ and that $H$ is a $T$-invariant subspace of $L$. By Theorem 2.3, there is a positive definite $T$-invariant quadratic form on $L$, relative to which $H$ has an orthogonal complement $H^+$ which is then also $T$-invariant. We define the orthogonal projection $p_H$ by $p_H: L \to L$, $p_H(L) = H$, $\ker p_H = H^+$. From Corollary 3.10 it follows quickly that $H$ is the fixed vector space for the action of $T$ on $L$ and from this it follows that the invariant complement $H^+$ is uniquely determined as the vector subspace spanned by the vector $\gamma \cdot v - v$, $v \in L$, $\gamma \in T$. Moreover, it follows that $p_H = \int_T \tau \, d\tau$ with normalized Haar measure on $T$ in the sense that

$$p_H(x) = \int_T \tau \cdot x \, dx \qquad \text{for} \quad x \in L.$$

4.7. Lemma. *If $H$ is a compactly embedded Cartan algebra of a finite dimensional Lie algebra $L$, and if $W$ is an invariant generating cone in $L$, then $H \cap W = p_H(W)$.*

*Proof.* Since $p_H$ is a projection onto $H$, the inclusion $H \cap W \subset p_H(W)$ is trivial. Now let $w \in W$. Then $p_H(w) \subset H$ by definition of $p_H$, and since $\tau \cdot w \in W$ for all inner automorphisms by invariance of $W$, this remains true for all $\tau \in L^{\mathrm{ad}}$ and thus, in particular for all $\tau \in T$. Since is a closed and convex, we conclude $p_H(w) = \int_T \tau \cdot w \, d\tau \in W$. Hence $p_H(w) \in H \cap W$. ▮

4.8. Lemma. *Under the circumstances of Lemma* 4.7, *if also $W \neq L$, then* $\operatorname{int}(W) \cap (L^{\mathrm{ad}}) \cdot H^+ \neq \varnothing$.

*Proof.* Since $W$ is invariant under $\operatorname{Inn}(L)$, hence also under the closure $L^{\mathrm{ad}} = (\operatorname{Inn}(L))^-$ in $gl(L)$, it suffices to show that $\operatorname{int}(W) \cap H^+ = \varnothing$. Suppose now that $x \in \operatorname{int}(W) \cap H^+$. Then $0 = p_H(x) \in p_H(\operatorname{int}(W)) = \operatorname{int}_H(p_H(W))$ (since $p_H$ is an open map!) $= \operatorname{int}_H(H \cap W)$ in view of Lemma 4.7. But now 0 is an inner point of the wedge $H \cap W$ in $H$, and thus $H \cap W = H$, i.e., $H \subset W \cap -W$. But the edge of the invariant cone $W$ is an ideal of $L$ which must be proper, since we assume $W \neq L$. However, no Cartan algebra can be contained in a proper ideal, since its image in the factor algebra must be a Cartan algebra. This contradiction proves the claim. ▌

4.9. LEMMA. *Under the circumstances of Lemma* 4.7, *the cone* $W \cap H$ *in* $H$ *is invariant under the action of the Weyl group.*

*Proof.* This is immediate since the action of the Weyl group on $H$ is induced by elements of $N(H, L)^*$ which leave $H$ invariant by definition of the normalizer and which leave $W$ invariant since $W$ is invariant. ▌

4.10. LEMMA. *Under the circumstances of Lemma* 4.7, *if* $W$ *is also pointed, then* $\operatorname{int}_H(W \cap H) \cap Z(K(H)) \neq \emptyset$ *where* $Z(K(H))$ *is the center of the unique maximal compactly embedded subalgebra* $K(H)$ *containing* $H$ (*see Theorem* 4.6).

*Proof.* The compact group $K(H)^{\mathrm{ad}}$ acts on the pointed generating cone $W$ and thus has a fixed vector $x \in \operatorname{int}(W)$ (e.g., the point $\int_{K(H)^{\mathrm{ad}}} \gamma \cdot w \, d\gamma$ for any preassigned point $w \in \operatorname{int}(W)$). Then $x \in Z(K(H))$. But also $Z(K(H)) \subset H$ since $H$ is a Cartan algebra of $K(H)$. Thus $x \in \operatorname{int}_L(W) \cap H \cap Z(K(H)) = \operatorname{int}_H(W \cap H) \cap Z(K(H))$ by our first observation in the proof of 4.6. ▌

Let us summarize the information obtained in the previous lemmas under the view point of finding necessary conditions for a cone in $H$ to be the intersection of an invariant cone in $L$ with $H$.

4.11. PROPOSITION. *Let* $W$ *be an invariant pointed generating cone in a finite dimensional Lie algebra* $L$; *let* $H$ *be a compactly embedded Cartan subalgebra and set* $C = W \cap H$. *We let* $p_H$ *be the vector space endomorphism given by* $p_H = \int_{H^{\mathrm{ad}}} \tau \, d\tau$. *Then we have the following conclusions*:

(i) $C = p_H(W)$.

(ii) $\operatorname{int}_H(C) \cap (L^{\mathrm{ad}})(\ker p_H) = \emptyset$.

(iii) *$C$ is invariant under the Weyl group.*

(iv) $\operatorname{int}_H(C) \cap Z(K(H)) \neq \emptyset$.

The conditions listed in Proposition 4.11 are by no means exhaustive. However, they are the ones we obtain without going into the root decomposition of $L$ with respect to $H$, and this will be discussed in [HHL89], Chap. III. Yet this is the point at which we wish to raise the question in which way we can produce from a pointed generating cone $C$ in a compactly embedded Cartan algebra of a Lie algebra an invariant cone $W$ in $L$—preferably in such a fashion that $C = H \cap W$. Again, a complete answer will have to be deferred to the monograph cited above. Nevertheless, a few observations fit well into our present frame work.

4.12. DEFINITION. For a cone $C$ in a compactly embedded Cartan algebra $H$ of a finite dimensional Lie algebra $L$ we write

$$C_L = \bigcap \{\gamma \cdot p_H^{-1}(C) : \gamma \in \operatorname{Inn}(L)\},$$

$$C^L = \overline{\operatorname{conv}}(\operatorname{Inn}(L) \cdot C),$$

where with $\overline{\operatorname{conv}}$ we indicate the formation of the closed convex hull.

4.13. PROPOSITION. (i) $C_L = \{x \in L : p_H((\operatorname{Inn}(L)) \cdot x) \subset C\}$.

(ii) *$C_L$ is an invariant cone in $L$ whose edge is the largest ideal of $L$ contained in $H^+ = \ker p_H$.*

(iii) $H \cap C_L = p_H(C_L) = \{c \in C : p_H((\operatorname{Inn}(L) \cdot c) \subset C\} \subset C$.

(i) $\operatorname{Inn}(L) \cdot C$ *has inner points if* $\operatorname{int}_H(C) \neq \varnothing$.

*Proof.* (i) By definition of $C_L$ we have $x \in C_L$ if and only if for each inner automorphism $\gamma$ the relation $x \in \gamma \cdot p_H^{-1}(C)$, i.e., $p_H(\gamma \cdot x) \in C$ holds. But this is exactly assertion (i).

(ii) The set $p_H^{-1}(C)$ is a cone, and then so are all the sets $\gamma \cdot p_H^{-1}(C)$, $\gamma \in \operatorname{Inn}(L)$. Hence their intersection is a cone. The invariance is clear, since each inner automorphism permutes the sets $\gamma \cdot p_H^{-1}(C)$. The edge of an invariant cone is always an ideal. Conversely, if $I$ is any ideal of $L$ in $\ker p_H$, then $\gamma \cdot I \subset \gamma \cdot p_H^{-1}(0) \subset \gamma \cdot p_H^{-1}(C)$, and thus $I$ is in the edge of $C_L$. Thus $H(C_L)$ is the largest ideal of $L$ contained in $\ker p_H$.

(ii) The first equality follows from Lemma 4.7. If $c \in H \cap C_L$, then in particular $c \in H \cap p_H^{-1}(C) = C$, and then the rest follows from (i).

(iv) Let $c$ be a regular point of $L$ in the interior of $C$; since the regular points of a Cartan subalgebra are dense, such a $c$ exists. Since $c$ is regular and $H$ is abelian, $H = \ker \operatorname{ad} c$. Now we apply Lemma 2.12 with $H$ in the place of $K$ and $c$ in place of $x$. This lemma allows us to conclude that $c$ is an inner point of $\operatorname{Inn}(L) \cdot C$. ∎

4.14. PROPOSITION. *$C^L$ is the smallest invariant cone in $L$ containing $C$.*

*Proof.* This is clear. ∎

The following summary then shows what we have to focus on when we deal in greater detail later with the question of which cones in $H$ determine a global invariant cone $W$ in $L$.

4.15. THEOREM (Construction theorem for invariant cones). *Let $H$ be a compactly embedded Cartan algebra in a finite dimensional real Lie algebra $L$, and let $C$ be a pointed generating cone in $H$. Then the following conditions are equivalent:*

(1) *There exists an invariant pointed generating cone $W$ in $L$ such that $C = H \cap W$.*

(2) $p_H((\mathrm{Inn}(L))\cdot C)\subset C$ (*i.e., each conjugacy class of an element $c\in C$ projects into C under $p_H$*), *and* $\ker p_H$ *contains no nonzero ideal of L.*

(3) $C=H\cap C_L$ *and* $\ker p_H$ *contains no nonzero ideal of L.*

*If these conditions are satisfied, then $W=C_L$.*

*Proof.* (1)$\Rightarrow$(3) We assume $C=H\cap W$ for an invariant pointed generating cone $W$ in $L$. Then for each inner automorphism $\gamma$ we have $W=\gamma\cdot W\subset\gamma\cdot p_H^{-1}p_H(W)=\gamma\cdot p_H^{-1}(H\cap W)=\gamma\cdot p_H^{-1}(C)$ in view of 4.11(i). Hence $W\subset C_L$ by the definition of $C_L$. By Lemma 4.7 we now observe $C=p_H(W)\subset p_L(C_L)=H\cap C_L\subset C$ in view of Proposition 4.13(iii), whence $C=H\cap C_L$. From the inclusion $W\subset C_L$ it follows that $C_L$ is generating.

We must show that $\ker p_H$ contains no nonzero ideal. In particular, this will show that $C_L$ is pointed by Proposition 4.13(ii), and then the Uniqueness Theorem 4.6 will also show $W=C_L$. Now suppose that $0\neq I$ is an ideal of $L$ contained in $H^+=\ker p_H$. The torus $H^{\mathrm{ad}}$ acts on $H^+$ without a fixed point (other than 0). Hence it acts on $I$ without a fixed point. Now let $J$ be a minimal nonzero subspace of $I$ invariant under the action of $H^{\mathrm{ad}}$. Then $\dim J=2$, since the torus $H^{\mathrm{ad}}$ cannot act on a one-dimensional real vector space in a nontrivial fashion. Then $J$ is an irreducible $H$-module under the adjoint action of $H$ on $H^+$. Since $H^{\mathrm{ad}}$ is a group of inner automorphisms, $[J,J]$ is $\mathrm{H}^{\mathrm{ad}}$ invariant. On the other hand, $\dim[J,J]\leqslant 1$ since $\dim J=2$. Hence $H^{\mathrm{ad}}$ leaves all points of $[J,J]$ elementwise fixed, whence $[J,J]\subset H\cap I\subset H\cap H^+=\{0\}$. Thus $J$ is an abelian ideal of $H+J$, and $H+J$ is a subalgebra. The representation $\rho\colon H\to gl(J)$, $\rho(x)=\mathrm{ad}\,x\mid J$ has a one-dimensional image; thus $\ker\rho=H_0$ is a hyperplane in $H$. Since $C$ is generating, we find an element $h\in\mathrm{int}(C)\setminus H_0$. Since $\mathrm{ad}\,h$ has a purely imaginary spectrum, then $\rho(h)$ has a purely imaginary spectrum. We conclude from this information the existence of two basis vectors $u$ and $v$ of $J$ with $\rho(h)(u)=v$ and $\rho(h)(v)=-u$. Thus the subalgebra $A=\mathrm{span}\{h,u,v\}$ is defined by the identities $[h,u]=v$, $[h,v]=-u$, $[u,v]=0$. Hence $A$ is the Lie algebra of the group of euclidean motions of the plane. Moreover, $W\cap A$ s a pointed, invariant cone in $A$, and since $h\in\mathrm{int}(W)$, we know that its interior in $A$ is nonempty and hence that it is generating. But all invariant nonsingleton cones of $A$ are known to contain $J=\mathrm{span}\{u,v\}$ (see, e.g., [HH85b]), and this is a contradiction to the fact that $C$ is pointed. This proves the claim that $\ker p_H$ does not contain any nontrivial ideal of $L$.

(3)$\Rightarrow$(2) Let $\gamma$ be an inner automorphism and $c\in C$. Then $c\in C_L$ by (3), and thus $c\in C_L\subset\gamma^{-1}p_H(C)$ whence $p_H(\gamma\cdot c)\in C$, which we had to show.

(2)$\Rightarrow$(1) We define $W=C_L$. Then $W$ is an invariant cone. By (2) we have $\mathrm{Inn}(L)\cdot C\subset p_H^{-1}(C)$ and thus $C\subset C_L$ by 4.13(i). In view of Proposition 4.13(iii) we now deduce $C = H \cap W$. By (2) and Proposition 4.13(ii) we know that $W$ is pointed. In order to conclude the proof we have to show that $W$ has interior points. By 4.13(iv), the interior of $\mathrm{Inn}(L) \cdot C$ is not empty. Hence $C^L$ has interior points. But since $W$ is invariant and contains $C$, from 4.14 we know that $C^L \subset W$. Hence $W$ has interior points. ∎

In Theorem 3.13 we have seen that every compactly embedded Cartan algebra uniquely determines a maximal compactly embedded subalgebra $K(H)$ with $H \subset K(H)$. If the presence of a compactly embedded Cartan algebra $H$ is due to the existence of an invariant pointed generating cone $W$ in $L$ then further information becomes available on $K(H)$. Quite generally, $K(H)$ as a compact Lie algebra is the direct sum of its center $Z(K(H))$ and its (semisimple) commutator algebra $K(H)'$. Naturally, $K(H)$ is contained in the centralizer $Z(Z(K(H)), L)$ of $Z(K(H))$ in $L$.

4.16. THEOREM. *Let $W$ be an invariant pointed generating cone in a finite dimensional real Lie algebra $L$ and let $H$ denote a compactly embedded Cartan algebra. Then the following propositions hold*:

(i) $Z(K(H)) \cap \mathrm{int}(W) \neq \varnothing$.

(ii) $K(H)' \cap W = \{0\}$

(iii) $Z(Z(K(H)), L) = K(H)$.

*Proof.* (i) The compact group $K(H)^{\mathrm{ad}}$ acts on $L$ leaving the pointed generating cone $W$ invariant. Hence there exists a fixed vector $x \in \mathrm{int}(W)$ for this action (see, e.g., [HH86b, Proposition 1.8]). Then $x \in Z(K(H))$, and (i) is proved.

(ii) The invariant cones in compact Lie algebras were completely classified in [HH86b]. In particular, [HH86b, Corollary 2.7] says that a compact semisimple Lie algebra cannot contain an invariant pointed cone.

(iii) The group $Z(Z(K(H)), L)^{\mathrm{ad}}$ leaves $W$ invariant and fixes at least one vector $x \in \mathrm{int}(W)$ according to (i) above. However, if $C$ denotes the subgroup of all $\gamma \in L^{\mathrm{ad}}$ fixing $x$, then $C$ is compact by Lemma 4.1. Thus $Z(Z(K(H)), L)^{\mathrm{ad}}$ is a compact group containing the maximal compact group $K(H)^{\mathrm{ad}}$ (see Theorem 1.10). Hence $Z(Z(K(H)), L)^{\mathrm{ad}} = K(H)^{\mathrm{ad}}$, whence $Z(Z(K(H)), L) = K(L)$ by Theorem 1.10. ∎

By Proposition 3.16, $K(H)$ also determines an orthogonal $K(H)$-module complement $P(H)$. On account of the unique $H^{\mathrm{ad}}$-module decomposition $L = H \oplus H^+$, we conclude from $H \subset K(H)$ that the $H^{\mathrm{ad}}$-submodule $P(H)$ is contained in $H^+$. From Theorem 4.15 we know that $H^+$ cannot contain any nonzero ideals of $L$ if $L$ contains a pointed generating invariant cone. The following is therefore an immediate consequence:

4.17. PROPOSITION. *Under the circumstances of Theorem* 4.16, $P(H)$ *cannot contain any nonzero ideals of* $L$.

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